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Linear Algebra (1)
A. Ram

Matrix monsters

$A \in M_{t\times s}(\mathbb{R})$ gives $T_A : \mathbb{R}^s \to \mathbb{R}^t$

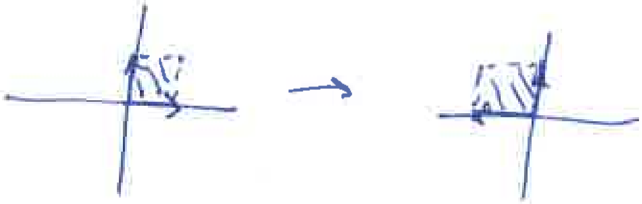
$$v \mapsto Av.$$

Example 5 $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ gives $T_A : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



T_A is reflection in the y-axis

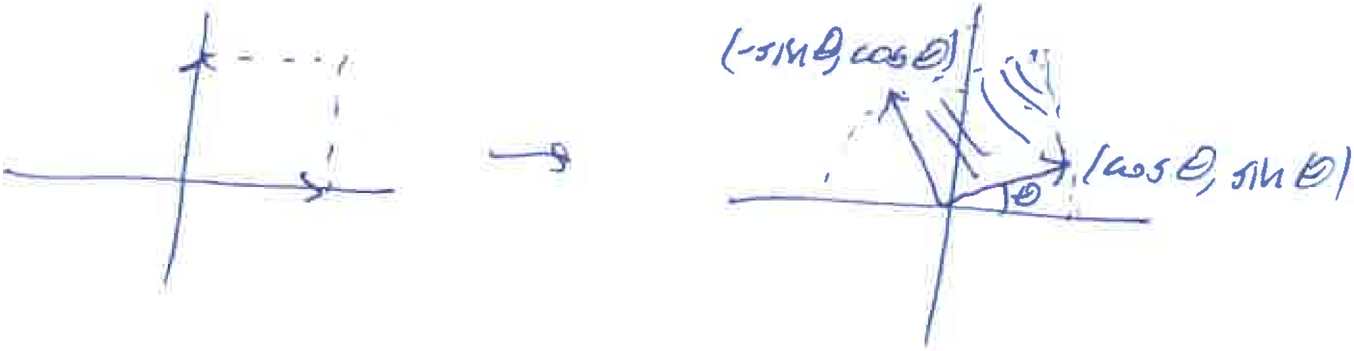
Example 7 $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ gives

$T_A : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

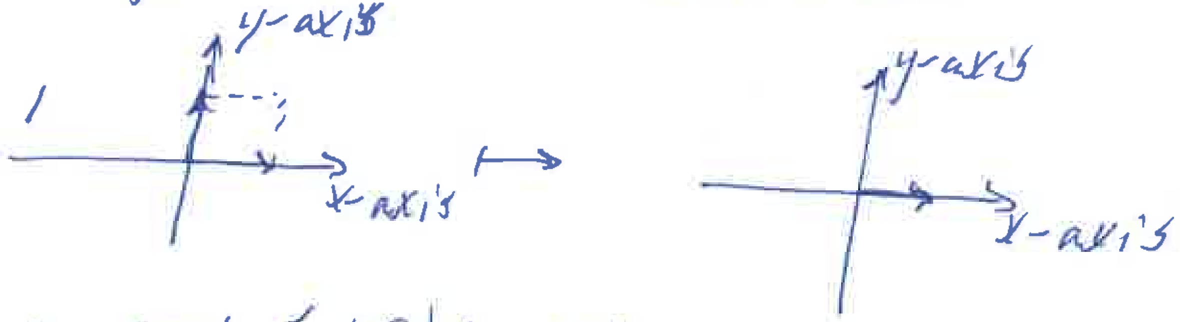
T_A is rotation in the angle θ.



<u>Example 7</u> $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ gives

$T_A : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad$ T_A is projection onto the x-axis

$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ 0 \end{pmatrix}$



$\ker(T_A) = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \middle| y \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\operatorname{im}(T_A) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

<u>Example 8</u> $A = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$ gives $T_A : \mathbb{R}^2 \to \mathbb{R}^2$

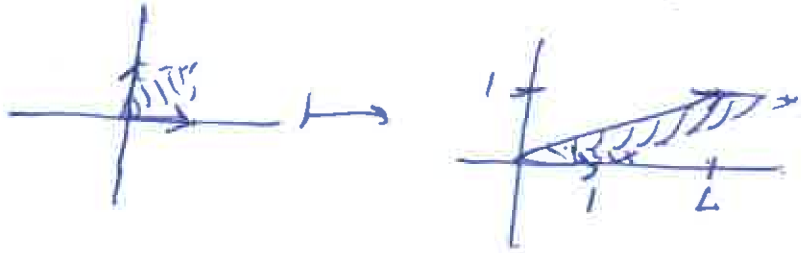
$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} c \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

<u>Example 9</u> $A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ gives $T_A : \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} c \\ 1 \end{pmatrix}$

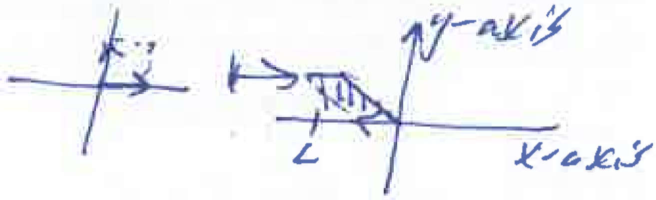


T_A is <u>shear along the x-axis with $c = 1$</u>.

<u>Example 10</u> $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ gives

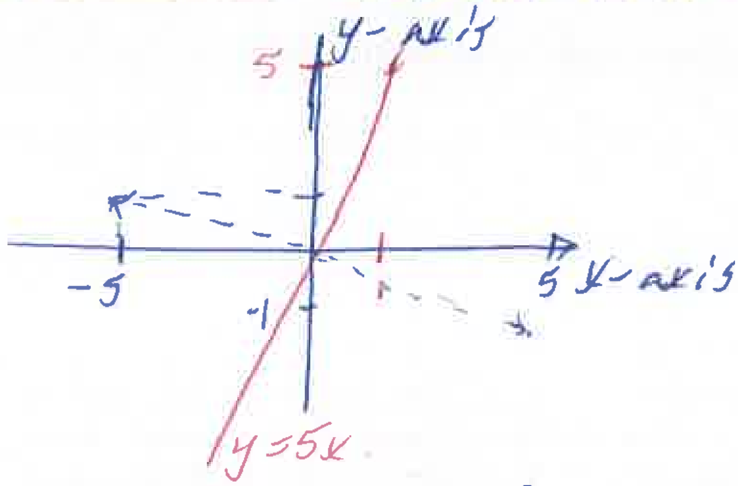
$$\begin{array}{ccccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \longmapsto & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \longmapsto & \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \longmapsto & \begin{pmatrix} 2 \\ 1 \end{pmatrix} & \longmapsto & \begin{pmatrix} -2 \\ 1 \end{pmatrix} \end{array}$$

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$



T_A is a shear along the x-axis followed by reflection in the y-axis

<u>Example 6 and 23</u> Reflection in the line $y = 5x$



Let $C = \{\langle 1,5\rangle, \langle 5,-1\rangle\}$.

The linear transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ -1 \end{pmatrix} \longmapsto \begin{pmatrix} -5 \\ +1 \end{pmatrix}$$

is reflection in the line $y=5$.

The matrix of T with respect to the basis $C = \{\langle 1,5\rangle, \langle 5,-1\rangle\}$ is

$$[T]_{CC} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The transition matrix from $B = \{\langle 1,0\rangle, \langle 0,1\rangle\}$ to C is

$$P_{CB} = \begin{pmatrix} 1 & 5 \\ 5 & -1 \end{pmatrix} \text{ since } \begin{matrix} \langle 1,5\rangle = 1\cdot\langle 1,0\rangle + 5\langle 0,1\rangle \\ \langle 5,-1\rangle = 5\cdot\langle 1,0\rangle + (-1)\langle 0,1\rangle. \end{matrix}$$

The inverse of P_{CB} is

$$P_{CB}^{-1} = \begin{pmatrix} -1 & -5 \\ -5 & 1 \end{pmatrix}\left(\frac{-1}{26}\right) = \begin{pmatrix} \frac{1}{26} & \frac{+5}{26} \\ \frac{+5}{26} & \frac{-1}{26} \end{pmatrix} = P_{BC}.$$

The matrix of T with respect to the basis B is

$$[T]_{BB} = P_{BC}\,[T]_{CC}\,P_{CB}$$

$$= \begin{pmatrix} \frac{1}{26} & \frac{5}{26} \\ \frac{5}{26} & \frac{-1}{26} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 5 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{26} & \frac{5}{26} \\ \frac{5}{26} & \frac{-1}{26} \end{pmatrix}\begin{pmatrix} 1 & 5 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} \frac{-24}{26} & \frac{10}{26} \\ \frac{10}{26} & \frac{24}{26} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{3} \end{pmatrix}$$