05.09.2023
Linear Algebra (1)
A. Ram

Example 12 Let T be the linear transformation $T: P_2 \to P_1$ given by

$$T(a_0 + a_1 x + a_2 x^2) = (a_0 + a_2) + a_0 x.$$

With respect to the bases

$B = \{1, x, x^2\}$ of P_2 (the source) and

$C = \{1, x\}$ of P_1 (the target)

the matrix of T is

$$[T]_{CB} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ since } \begin{aligned} T(1) &= 1 + 1 \cdot x \\ T(x) &= 0 + 0x \\ T(x^2) &= 1 + 0x. \end{aligned}$$

With respect to the basis

$B = \{1, x, x^2\}$ of P_2 (the source) and

$D = \{2, 3x\}$ of P_1 (the target)

the matrix of T is

$$[T]_{DB} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \end{pmatrix} \text{ since } \begin{aligned} T(1) &= \tfrac{1}{2} \cdot 2 + \tfrac{1}{3} \cdot 3x \\ T(x) &= 0 \cdot 2 + 0 \cdot 3x \\ T(x^2) &= \tfrac{1}{2} \cdot 2 + 0 \cdot 3x. \end{aligned}$$

Then

$\ker(T) = \{a_1 x \mid a_1 \in \mathbb{R}\} = \mathrm{span}\{x\}$ and

$\mathrm{im}(T) = \{c_0 + c_1 x \mid c_0, c_1 \in \mathbb{R}\} = P_1.$

So $\dim(\ker(H)) + \dim(\operatorname{im}(H)) = 1 + 2 = 3$

and $3 = \dim(P_2) = \dim(\operatorname{source}(H))$.

Example 34 Topic 4

Let $v = \langle 1,5 \rangle \in \mathbb{R}^2$.

The coordinate vector of v with respect to the basis $B = \{\langle 1,0\rangle, \langle 0,1\rangle\}$ is

$$[v]_B = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \text{ since } v = \langle 1,5\rangle = 1\cdot\langle 1,0\rangle + 5\cdot\langle 0,1\rangle$$

The coordinate vector of v with respect to the basis $C = \{\langle 2,1\rangle, \langle -1,1\rangle\}$ is

$$[v]_C = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

since $\langle 1,5\rangle = \cancel{3\cdot\langle 2,1\rangle + 2\langle -1,1\rangle}$

$$= 2\cdot\langle 2,1\rangle + 3\langle -1,1\rangle$$

Example 35 Topic 4

The coordinate vector of $p = 2 + 7x - 9x^2$ with respect to the basis

$B = \{2, 2x, -3x^2\}$ of P_2 is

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③

$$[p]_B = \begin{pmatrix} 1 \\ 14 \\ 3 \end{pmatrix}$$

since

$$2 + 7x - 9x^2 = 1 \cdot 2 + 14 \cdot \tfrac{1}{2}x + 3 \cdot (-3x^2)$$

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Linear Algebra ①
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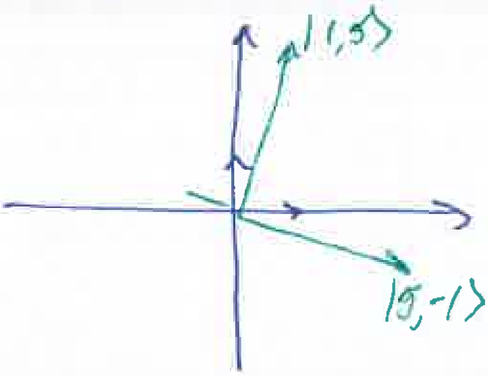
<u>Example</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $y = 5x$.



$B = \{\langle 1,0\rangle, \langle 0,1\rangle\}$

$C = \{\langle 1,5\rangle, \langle 5,-1\rangle\}$

$$[T]_{CC} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$P_{CB} = \begin{pmatrix} 1 & 5 \\ 5 & -1 \end{pmatrix} \qquad P_{BC} = \begin{pmatrix} \frac{1}{26} & \frac{5}{26} \\ \frac{5}{26} & \frac{-1}{26} \end{pmatrix}$$

then

$$[T]_{BB} = P_{BC}[T]_{CC}P_{CB}$$

$$= \begin{pmatrix} \frac{1}{26} & \frac{5}{26} \\ \frac{5}{26} & \frac{-1}{26} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 5 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{26} & \frac{-5}{26} \\ \frac{5}{26} & \frac{1}{26} \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} \frac{-24}{26} & \frac{10}{26} \\ \frac{10}{26} & \frac{24}{26} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}$$

08.04.2023
Linear Algebra (2)
A. Ram

Let V be an $\mathbb{R}$-vector space.

A <u>subspace of V</u> is a subset W of V such that

(1) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$

(2) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.

(3) $0 \in W$

Let X and Y be $\mathbb{R}$-vector spaces.

A <u>linear transformation</u> from X to Y is a function $T: X \to Y$ such that

(1) If $x_1, x_2 \in X$ then $T(x_1 + x_2) = T(x_1) + T(x_2)$

(2) If $x \in X$ and $c \in \mathbb{R}$ then $T(cx) = cT(x)$.

The <u>kernel and image of T</u> are

$$\ker(T) = \{x \in X \mid T(x) = 0\}$$

$$\mathrm{im}(T) = \{T(x) \mid x \in X\}.$$

<u>Proposition</u> Let V be a vector space.

(1) If $v \in V$ then $0 \cdot v = 0$.

(2) If $v \in V$ then $(-1) \cdot v = -v$.

08.09.2023
Linear Algebra (3)
A. Ram

Let $f: S \to T$ be a function.

The function f is <u>injective</u> if f satisfies:

if $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$ then $s_1 = s_2$.

The function f is <u>surjective</u> if f satisfies:

if $t \in T$ then there exists $s \in S$ such that $f(s) = t$.

<u>Theorem</u> Let $T: X \to Y$ be a linear transformation.

(1) $\ker(T)$ is a subspace of X

(2) $\mathrm{im}(T)$ is a subspace of Y.

(3) T is injective if and only if $\ker(T) = 0$.

(4) T is surjective if and only if $\mathrm{im}(T) = 0$.

<u>Proof</u> (3) To show: (a) If T is injective then $\ker(T) = 0$

(b) If $\ker(T) = 0$ then T is injective.

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Linear Algebra (4)
A. Ram

(3a) Assume $T: X \to Y$ is injective.

To show: $\ker(T) = 0$.

To show: If $x \in X$ and $T(x) = 0$ then $x = 0$.

Assume $x \in X$ and $T(x) = 0$.

Then $T(x) = T(0)$.

Since T is injective and $T(x) = T(0)$ then $x = 0$.

(3b) Assume $\ker(T) = 0$.

To show: $T: X \to Y$ is injective.

To show: If $x_1, x_2 \in X$ and $T(x_1) = T(x_2)$ then $x_1 = x_2$.

Assume $x_1, x_2 \in X$ and $T(x_1) = T(x_2)$.

To show: $x_1 = x_2$.

Since $0 = T(x_1) - T(x_2) = T(x_1 - x_2)$

then $T(0) = T(x_1 - x_2)$.

So $x_1 - x_2 = 0$.

So $x_1 = x_2$.

So $T: X \to Y$ is injective if and only if $\ker(T) = 0$.

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Linear Algebra (5)
A. Ram

(1) To show: $\ker(T)$ is a subspace of X.

To show: (1a) $0 \in \ker(T)$

(1b) If $x_1, x_2 \in \ker(T)$ then $x_1 + x_2 \in \ker(T)$

(1c) If $x \in \ker(T)$ and $c \in \mathbb{R}$ then $cx \in \ker(T)$.

(1a) Since $T(0) = 0$ then $0 \in \ker(T)$.

(1b) Assume $x_1, x_2 \in \ker(T)$.

To show: $x_1 + x_2 \in \ker(T)$

Since $x_1, x_2 \in \ker(T)$ then $T(x_1) = 0$ and $T(x_2) = 0$.
Since T is a linear transformation then
$$T(x_1 + x_2) = T(x_1) + T(x_2) = 0 + 0$$

So $x_1 + x_2 \in \ker(T)$.

(1c) Assume $x \in \ker(T)$ and $c \in \mathbb{R}$.

To show: $cx \in \ker(T)$.

Since $x \in \ker(T)$ then $T(x) = 0$.
Then, since T is a linear transformation,
$$T(cx) = cT(x) = c \cdot 0 = 0.$$

So $cx \in \ker(T)$.

So $\ker(T)$ is a subspace of X.