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Linear Algebra ①

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<u>Example 15</u> Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(|x,y,z\rangle) = |2x-y, y+z\rangle.$$

Then

$$T(|1,0,0\rangle) = |2\cdot 1-0, 0+0\rangle = |2,0\rangle$$
$$T(|0,1,0\rangle) = |2\cdot 0-1, 1+0\rangle = |-1,1\rangle$$
$$T(|0,0,1\rangle) = |2\cdot 0-0, 0+1\rangle = |0,1\rangle$$

and so the matrix of T with respect to

$B = \{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\}$ a basis of $\mathbb{R}^3$

and $C = \{|1,0\rangle, |0,1\rangle\}$ a basis of $\mathbb{R}^2$, is

$$[T]_{CB} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then

$$\ker(T) = \{|x,y,z\rangle \mid T(|x,y,z\rangle) = |0,0\rangle\}$$
$$= \{|x,y,z\rangle \mid |2x-y, y+z\rangle = |0,0\rangle\}$$
$$= \left\{|x,y,z\rangle \,\middle|\, \begin{matrix} 2x-y=0 \\ y+z=0 \end{matrix}\right\}$$
$$= \left\{|x,y,z\rangle \,\middle|\, \begin{matrix} 2x=y \\ y=-z \end{matrix}\right\} = \left\{|x,y,z\rangle \,\middle|\, \begin{matrix} 2x=-z \\ y=-z \end{matrix}\right\}$$
$$= \{|x,y,z\rangle \mid x = -\tfrac{1}{2}z,\ y=-z\}$$
$$= \mathrm{span}\{|-\tfrac{1}{2}, -1, 1\rangle\}.$$

and

$$\mathrm{im}(T) \supseteq \{T(\langle \tfrac{1}{2},0,0\rangle), T(\langle 0,0,1\rangle)\}$$
$$= \{\langle 2\cdot\tfrac{1}{2}-0, 0+0\rangle, \langle 0-0, 0+1\rangle\}$$
$$= \{\langle 1,0\rangle, \langle 0,1\rangle\}.$$

Since $\mathrm{im}(T)$ is a subspace of $\mathbb{R}^2$ then

$$\mathrm{im}(T) \supseteq \mathrm{span}\{\langle 1,0\rangle, \langle 0,1\rangle\} = \mathbb{R}^2.$$

Since $\mathrm{im}(T) \subseteq \mathbb{R}^2$ then $\mathrm{im}(T) = \mathbb{R}^2$.

Note that $\mathrm{im}(T) = \mathrm{span}\left\{\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$

$$= \mathrm{colspace}\begin{pmatrix}2 & -1 & 0\\ 0 & 1 & 1\end{pmatrix}.$$

<u>Examples 16 and 17</u> Let $T: P_2 \to P_1$ be defined by

$$T(a_0+a_1x+a_2x^2) = (a_0-a_1+a_2)(1+2x).$$

Since
$$T(1) = T(1+0x+0x^2) = (1-0+0)(1+2x) = 1+2x$$
$$T(x) = T(0+1\cdot x+0x^2) = (0-1+0)(1+2x) = -(1+2x)$$
$$T(x^2) = T(0+0\cdot x+1\cdot x^2) = (0-0+1)(1+2x) = 1+2x$$

then the matrix of T with respect to

$B = \{1, x, x^2\}$ a basis of P_2

and $C = \{1, x\}$ a basis of P_1

15.09.2023 ③
Linear Algebra
A. Ram

is $[T]_{CB} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \end{pmatrix}$

Then

$$\ker(T) = \{ a_0 + a_1 x + a_2 x^2 \mid T(a_0 + a_1 x + a_2 x^2) = 0 \}$$

$$= \{ a_0 + a_1 x + a_2 x^2 \mid (a_0 - a_1 + a_2)(1 + 2x) = 0 \}$$

$$= \{ a_0 + a_1 x + a_2 x^2 \mid a_0 - a_1 + a_2 = 0 \}$$

$$= \{ a_0 + a_1 x + a_2 x^2 \mid a_0 = a_1 - a_2 \}$$

$$= \{ (a_1 - a_2) + a_1 x + a_2 x^2 \mid a_1, a_2 \in \mathbb{R} \}$$

$$= \mathrm{span}\{ 1 + x, -1 + x^2 \}$$

and $\quad \mathrm{im}(T) = \mathrm{span}\{(1+2x)\}$

since $1 + 2x = T(1) \in \mathrm{im}(T)$ and

$\mathrm{im}(T)$ is a subspace then $\mathrm{im}(T) \supseteq \mathrm{span}\{1+2x\}$

and $\mathrm{im}(T) \subseteq \mathrm{span}\{1+2x\}$ since

$(a_0 - a_1 + a_2)(1 + 2x) \in \mathrm{span}\{1 + 2x\}$

for $a_0, a_1, a_2 \in \mathbb{R}$.

So

$\ker(T) = \mathrm{span}\{1 + x, -1 + x^2\}$ and $\mathrm{im}(T) = \mathrm{span}\{1 + 2x\}$

<u>Example 13</u> Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ have matrix

$$[T]_{SS} = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix}$$

with respect to the basis $S = \{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\}$

Then

$$T(|1,1,0\rangle) = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} = 6|1,1\rangle$$

$$T(|1,-1,0\rangle) = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix} = 4\cdot|1,-1\rangle$$

$$T(|1,-1,-2\rangle) = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = 2|1,1\rangle + 2|1,-1\rangle.$$

So the matrix of T with respect to

$$B = \{|1,1,0\rangle, |1,-1,0\rangle, |1,-1,-2\rangle\} \text{ of } \mathbb{R}^3$$

and $C = \{|1,1\rangle, |1,-1\rangle\}$ of $\mathbb{R}^2$

is

$$[T]_{CB} = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 4 & 2 \end{pmatrix}.$$

Examples 20 and 21 Let

$B = \{\langle 1,1\rangle, \langle -1,1\rangle\}$ a basis of $\mathbb{R}^2$

and $S = \{\langle 1,0\rangle, \langle 0,1\rangle\}$ a basis of $\mathbb{R}^2$.

Then the change of basis matrix between B and S is

$$P_{SB} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } P_{BS} = P_{SB}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Let $[v]_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $v = 1 \cdot \langle 1,1\rangle + 1 \cdot \langle -1,1\rangle = \langle 0,2\rangle$

$$[v]_S = P_{SB}[v]_B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Let $[w]_S = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Then $w = 0 \cdot \langle 1,0\rangle + 2 \cdot \langle 0,1\rangle = \langle 0,2\rangle$

and $[w]_B = P_{BS}[w]_S = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Example 22 Let

$$B = \{ |1,2\rangle, |1,1\rangle \} \text{ a basis of } \mathbb{R}^2$$

and $$C = \{ |-3,4\rangle, |1,-1\rangle \} \text{ a basis of } \mathbb{R}^2$$

and $$S = \{ |1,0\rangle, |0,1\rangle \} \text{ a basis of } \mathbb{R}^2.$$

Then the change of basis matrices are

$$P_{SB} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad P_{SC} = \begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix}$$

$$P_{BS} = -\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$P_{CS} = -\begin{pmatrix} -1 & -1 \\ -4 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}$$

and

$$P_{CB} = P_{CS} P_{SB} = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 10 & 7 \end{pmatrix}.$$