01.10.2013 (1)
Linear Algebra
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Let $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$. Let

$\bar{\ }:\mathbb{C}\to\mathbb{C}$ be given by $\overline{a+bi}=a-bi$

$\bar{\ }:\mathbb{R}\to\mathbb{R}$ be given by $\bar{a}=a$.

Let V be an $\mathbb{F}$-vector space. An <u>inner product on V</u> is a function

$$\begin{matrix} V\times V\to\mathbb{F} \\ (u,v)\mapsto\langle u,v\rangle \end{matrix}$$ such that

(1) If $u,v\in V$ then $\langle u,v\rangle=\overline{\langle v,u\rangle}$,

(2) If $u,v\in V$ and $\alpha\in\mathbb{F}$ then $\langle \alpha u,v\rangle=\alpha\langle u,v\rangle$,

(3) If $u,v,w\in V$ then $\langle u+v,w\rangle=\langle u,w\rangle+\langle v,w\rangle$,

(4) If $u\in V$ then $\langle u,u\rangle\in\mathbb{R}_{\geq 0}$

(5) If $u\in V$ and $\langle u,u\rangle=0$ then $u=0$.

Let V be an $\mathbb{F}$-vector space with an inner product.

<u>Length</u>: $\|\ \|:V\to\mathbb{R}_{\geq 0}$ given by $\|u\|^2=\langle u,u\rangle$.

<u>Distance</u>: $d: V\times V\to\mathbb{R}_{\geq 0}$ given by

$$d(u,v)=\|v-u\|.$$

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Angle: $\theta : V\times V \to \mathbb{R}_{[0,\pi]}$ given by

$$\cos(\theta(u,v)) = \frac{\mathrm{Re}\langle u,v\rangle}{\|u\|\cdot\|v\|}$$

Vectors u, v are orthogonal if $\langle u,v\rangle = 0$.

Standard inner products

(1) $\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$

$(u,v) \mapsto \langle u|v\rangle$ given by

$$\langle u_1,u_2,\dots u_n | v_1,v_2,\dots,v_n\rangle = u_1v_1+\dots+u_nv_n$$

(2) $\mathbb{C}^n\times\mathbb{C}^n \to \mathbb{R}$ given by

$(u,v) \mapsto \langle u|v\rangle$

$$\langle u_1,u_2,\dots u_n | v_1,v_2,\dots,v_n\rangle = u_1\bar{v}_1+\dots+u_n\bar{v}_n$$

(3) $\mathcal{P}_n\times\mathcal{P}_n \to \mathbb{F}$ given by

$$\langle a_0+a_1x+\dots+a_nx^n | c_0+c_1x+\dots+c_nx^n\rangle$$

$$= \int_0^1 (a_0+a_1x+\dots+a_nx^n)(\bar{c}_0+\bar{c}_1x+\dots+\bar{c}_nx^n)\,dx.$$

Example 5 $u=(1+i,1-i)$
$v=(i,1)$

in $\mathbb{C}^2$ with the standard inner product.

$\langle u|v\rangle = \langle 1+i,1-i \,|\, i,1\rangle = (1+i)\bar{i} + (1-i)\cdot\bar{1}$

$= (1+i)(-i) + 1-i = -i+1+1-i = 2-2i,$

$\langle v|u\rangle = \langle i,1 \,|\, 1+i,1-i\rangle = i\overline{(1+i)} + 1\cdot\overline{(1-i)}$

$= i(1-i) + 1+i = i+1+1+i = 2+2i,$

$\langle u,u\rangle = \langle 1+i,1-i \,|\, 1+i,1-i\rangle = (1+i)\overline{(1+i)} + (1-i)\overline{(1-i)}$

$= (1+i)(1-i) + (1-i)(1+i) = 1+1+1+1 = 4$

$d(u,v) = \sqrt{\langle 1,-i \,|\, 1,-i\rangle} = \sqrt{1\cdot\bar{1} + (-i)\overline{(-i)}} = \sqrt{1+1} = \sqrt{2}$

$\cos(\theta(u,v)) = \dfrac{\mathrm{Re}\langle u|v\rangle}{\|u\|\cdot\|v\|} = \dfrac{\mathrm{Re}(2-2i)}{2\cdot\sqrt{2}} = \cancel{\frac{1}{\sqrt{2}}(1-i)} = \dfrac{2}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}}.$

$\langle v|v\rangle = \langle i,1 \,|\, i,1\rangle = i\cdot\bar{i} + 1\cdot\bar{1} = i(-i)+1 = 1+1 = 2$

So $\theta(u,v) = \dfrac{\pi}{4}.$

<u>Example 7</u> $u = 1$ and $v = x$ in P_2 with the standard inner product.

$$\langle u|u\rangle = \langle 1|1\rangle = \int_0^1 dx = x\Big]_0^1 = 1$$

$$\langle v|v\rangle = \langle x|x\rangle = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

$$\langle u|v\rangle = \langle 1|x\rangle = \int_0^1 x\,dx = \frac{1}{2}x^2\Big]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$d(u,v) = \sqrt{\langle x-1|x-1\rangle} = \sqrt{\int_0^1 (x-1)^2\,dx}$$

$$= \sqrt{\frac{1}{3}(x-1)^3\Big]_0^1} = \sqrt{0 - \frac{1}{3}(-1)^3} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}.$$

$$\cos(\theta(u,v)) = \frac{\mathrm{Re}(\langle u|v\rangle)}{\|u\|\cdot\|v\|} = \frac{\frac{1}{2}}{1\cdot\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{2}$$

so that $\theta(u,v) = \frac{\pi}{6}$.