<u>Example</u> P_2 has basis $B = \{1, x, x^2\}$.

Since the standard inner product $\langle \mid \rangle : P_2 \times P_2 \to \mathbb{R}$ has

$$\langle p, q \rangle = \int_0^1 pq \, dx$$

then

$$\langle 1|1\rangle = 1, \quad \langle 1|x\rangle = \frac{1}{2}, \quad \langle 1|x^2\rangle = \frac{1}{3}$$

$$\langle x|1\rangle = \frac{1}{2} \quad \langle x|x\rangle = \frac{1}{3} \quad \langle x|x^2\rangle = \frac{1}{4}$$

$$\langle x^2|1\rangle = \frac{1}{3} \quad \langle x^2|x\rangle = \frac{1}{4} \quad \langle x^2|x^2\rangle = \frac{1}{5}$$

then the Gram matrix of $\langle \mid \rangle$ with respect to B is

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

If $u = 7 + 3x + 2x^2$ and $v = 5 + x^2$

then

$$[u]_B = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad [v]_B = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

Then

$$\langle u|v\rangle = \int_0^1 (7 + 3x + 2x^2)(5 + x^2)\, dx$$

$$= \int_0^1 (35 + 7x^2 + 15x + 3x^3 + 10x^2 + 2x^4)\,dx$$

$$= 35 + \frac{7}{3} + \frac{15}{2} + \frac{3}{4} + \frac{10}{3} + \frac{2}{5}$$

and

$$[u]_B^t A [v]_B = (7\ 3\ 2) \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

$$= (7\ 3\ 2) \begin{pmatrix} 5 + \frac{1}{3} \\ \frac{5}{2} + \frac{1}{4} \\ \frac{5}{3} + \frac{1}{5} \end{pmatrix}$$

$$= 35 + \frac{7}{3} + \frac{15}{2} + \frac{3}{4} + \frac{10}{3} + \frac{2}{5}.$$

So $\langle u|v\rangle = [u]_B^t A [v]_B$.

Let V be an $\mathbb{F}$-vector space with an inner product $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$.

Let $u, v \in V$.

The vectors u and v are <u>orthogonal</u> if $\langle u, v\rangle = 0$.

06.10.2023
Linear algebra (3)
A. Ram

An **orthogonal sequence** is a sequence $(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k)$ of vectors in V such that

if $i, j \in \{1, \dots, k\}$ and $i \neq j$

then $\langle \vec{v}_i, \vec{v}_j \rangle = 0$.

An **orthonormal sequence** is an orthogonal sequence $(\vec{v}_1, \dots, \vec{v}_k)$ such that

if $i \in \{1, \dots, k\}$ then $\langle \vec{v}_i, \vec{v}_i \rangle = 1$.

Example Let $\langle\ ,\ \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be given by

$$\langle (u_1, u_2, u_3), (v_1, v_2, v_3) \rangle = u_1 v_1 + 2 u_2 v_2 + u_3 v_3.$$

Let

$$S = \{(1,1,1), (1,-1,1), (1,0,-1)\}$$

then

$$\langle (1,1,1), (1,-1,1) \rangle = 1 - 2 + 1 = 0,$$

$$\langle (1,1,1), (1,0,-1) \rangle = 1 + 0 - 1 = 0,$$

$$\langle (1,-1,1), (1,0,-1) \rangle = 1 + 0 - 1 = 0.$$

So S is an orthogonal sequence in V.

06.10.2023
Linear algebra (4)
A. Ram

Let $b_1 = \frac{1}{\|\vec{u}_1\|}\vec{u}_1$, where $\vec{u}_1 = (1,1,1)$

$b_2 = \frac{1}{\|\vec{u}_2\|}\vec{u}_2$, where $\vec{u}_2 = (1,-1,1)$

$b_3 = \frac{1}{\|\vec{u}_3\|}\vec{u}_3$, where $\vec{u}_3 = (1,0,-1)$.

Then

$b_1 = \frac{1}{2}(1,1,1)$ since $\langle (1,1,1),(1,1,1)\rangle = 4$,

$b_2 = \frac{1}{2}(1,-1,1)$ since $\langle (1,-1,1),(1,-1,1)\rangle = 4$,

$b_3 = \frac{1}{\sqrt{2}}(1,0,-1)$ since $\langle (1,0,-1),(1,0,-1)\rangle = 2$.

and $\{b_1,b_2,b_3\}$ is an orthonormal sequence in V.

Let $x = (1,1,-1)$ and $c_1,c_2,c_3 \in \mathbb{R}$ such that

$$x = c_1 b_1 + c_2 b_2 + c_3 b_3.$$

Then

$$c_1 = c_1\langle b_1,b_1\rangle + 0 + 0 = \langle c_1 b_1 + c_2 b_2 + c_3 b_3, b_1\rangle$$
$$= \langle x, b_1\rangle = \langle (1,1,-1), \tfrac{1}{2}(1,1,1)\rangle = \tfrac{1}{2}(1+2-1)$$
$$= 1.$$

$$c_2 = c_2\langle b_2,b_2\rangle = \langle c_1 b_1 + c_2 b_2 + c_3 b_3, b_2\rangle$$
$$= \langle x, b_2\rangle = \langle (1,1,-1), \tfrac{1}{2}(1,-1,1)\rangle = \tfrac{1}{2}(1-2-1) = -1.$$

06.10.2023 (5)
Linear algebra
A. Ram

$$c_3 = \langle x, b_3 \rangle = \langle (1,1,-1), \tfrac{1}{\sqrt{2}}(1,0,-1) \rangle$$
$$= \tfrac{1}{\sqrt{2}}(1+0+1) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

So
$$x = \langle x, b_1 \rangle b_1 + \langle x, b_2 \rangle b_2 + \langle x, b_3 \rangle b_3$$
$$= 1 \cdot b_1 + (-1) \cdot b_2 + \cancel{0 \cdot b_3}\ \sqrt{2}\, b_3$$

Check: $b_1 - b_2 + \sqrt{2}\, b_3 = \tfrac{1}{2}(1,1,1) - \tfrac{1}{2}(1,-1,1) + \tfrac{\sqrt{2}}{\sqrt{2}}(1,0,-1)$
$$= (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) - (\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}) + (1,0,-1)$$
$$= (0,1,0) + (1,0,-1) = (1,1,-1) = x.$$