19.5.3 Algebraic, transcendental, normal, separable and perfect

26. Show that $\alpha = 2\pi i$ is algebraic over $\mathbb{R}$ and transcendental over $\mathbb{Q}$.
27. Give an example of a field of characteristic p such that the Frobenius map is not an automorphism.
28. Let $\mathbb{E} \supseteq \mathbb{F}$ be an inclusion of fields and let $\alpha \in \mathbb{E}$.
    (a) Carefully define what it means for α to be algebraic over $\mathbb{F}$.
    (b) Carefully define what it means for α to be transcendental over $\mathbb{F}$.
    (c) Carefully define what it means for α to be separable over $\mathbb{F}$.
    (d) Carefully define what it means for α to be normal over $\mathbb{F}$.
    (e) Carefully define what it means for α to be Galois over $\mathbb{F}$.
29. Let $\mathbb{E} \supseteq \mathbb{F}$ be an inclusion of fields and let $\alpha \in \mathbb{E}$. Show that if α is algebraic over $\mathbb{F}$ then $\mathbb{F}(\alpha)$ is a finite extension of $\mathbb{F}$.
30. Let $\mathbb{E} \supseteq \mathbb{F}$ be an inclusion of fields and let $\alpha \in \mathbb{E}$. Show that if α is transcendental over $\mathbb{F}$ then $\mathbb{F}(\alpha)$ is not a finite extension of $\mathbb{F}$.
31. Let $\mathbb{E} \supseteq \mathbb{F}$ be an inclusion of fields and let $\alpha \in \mathbb{E}$.
    (a) Carefully define what it means for α to be algebraic over $\mathbb{F}$.
    (b) Carefully define what it means for α to be transcendental over $\mathbb{F}$.
    (c) Carefully define what it means for α to be separable over $\mathbb{F}$.
    (d) Carefully define what it means for α to be normal over $\mathbb{F}$.
    (e) Carefully define what it means for α to be Galois over $\mathbb{F}$.
32. Let $\mathbb{E} \supseteq \mathbb{F}$ be an inclusion of fields.
    (a) Carefully define what it means for $\mathbb{E}$ to be a finite extension of $\mathbb{F}$.
    (b) Carefully define what it means for $\mathbb{E}$ to be an algebraic extension of $\mathbb{F}$.
    (c) Carefully define what it means for $\mathbb{E}$ to be a separable extension of $\mathbb{F}$.
    (d) Carefully define what it means for $\mathbb{E}$ to be a normal extension of $\mathbb{F}$.
    (e) Carefully define what it means for $\mathbb{E}$ to be a Galois extension of $\mathbb{F}$.
33. Determine which properties $\mathbb{R}/\mathbb{Q}$ and $\mathbb{C}/\mathbb{Q}$ and $\mathbb{R}/\mathbb{Q}$ have (finite, algebraic, separable, normal, Galois).
34. Show that $2^{\frac{1}{3}}$ is algebraic over $\mathbb{Q}$ and find the minimal polynomial.
35. Show that $\sqrt{3} + \sqrt{2}$ is algebraic over $\mathbb{Q}$ and find the minimal polynomial.
36. Show that $\frac{1}{2}(\sqrt{5} + 1)$ is algebraic over $\mathbb{Q}$ and find the minimal polynomial.
37. Show that $\frac{1}{2}(\sqrt{3} - 1)$ is algebraic over $\mathbb{Q}$ and find the minimal polynomial.
38. Prove that
$$\sum_{n \in \mathbb{Z}_{>\geq 0}} 10^{-n!} \qquad \text{is transcendental over } \mathbb{Q}.$$

39. Prove that e is transcendental over $\mathbb{Q}$.

40. Prove that π is transcendental over $\mathbb{Q}$.

41. Let $\mathbb{K} \supseteq \mathbb{F}$ be an extension. Show that the set of elements of $\mathbb{K}$ that are algebraic over $\mathbb{F}$ is a subfield of $\mathbb{K}$.

42. Let $\mathbb{F}$ be a field. Show that if α is algebraic over $\mathbb{F}$ then $\mathbb{F}[\alpha]$ is a field.

43. Let $\mathbb{K} \supseteq \mathbb{F}$ be a field extension and let $\alpha \in \mathbb{K}$. Let $f \in \mathbb{F}[x]$ be the minimal polynomial of α. Show that f is irreducible, that $\mathbb{F}(\alpha) = \mathbb{F}[\alpha]$ and that $\mathbb{F}(\alpha)$ has $\mathbb{F}$-basis $\{1, \alpha, \cdots, \alpha^{n-1}\}$, where $n = \deg(f)$.

44. The "Theorem of Louiville" states that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded then f is constant. Use Louiville's theorem to prove that $\mathbb{C}$ is algebraically closed. (Be sure to give a careful definition of $\mathbb{C}$.)

45. Let $\mathbb{F}$ be a field. Carefully define *algebraically closed* and *the algebraic closure of* $\mathbb{F}$. Show that the algebraic closure of $\mathbb{F}$ exists, is unique, is algebraic over $\mathbb{F}$ and is algebraically closed.

46. Show that $\overline{\mathbb{Q}} \neq \mathbb{C}$.

47. Let $\mathbb{F}$ be a field and let $J \subseteq \mathbb{F}[x]$. Carefully define *the splitting field of J over* $\mathbb{F}$. Show that the splitting field of J over $\mathbb{F}$ exists, is unique, and is algebraic over $\mathbb{F}$.

48. Suppose that E and K are two extensions of F and let $a \in E$ and $b \in K$ be algebraic over F. Prove that $m_{a,F} = m_{b,F}$ if and only if there exists an isomorphism $\varphi : F(a) \to F(b)$ such that $\varphi(a) = b$ and $\varphi|_F = \mathrm{id}_F$.

49. Let $E = \{a \in R \mid a \text{ is algebraic over } \mathbb{Q}\}$. Show that E is an algebraic extension of $\mathbb{Q}$ but is not a finite extension of $\mathbb{Q}$.

50. Show that the set of algebraic numbers (over $\mathbb{Q}$) in $\mathbb{R}$ forms a subfield of $\mathbb{R}$.

51. Show that every finite extension is algebraic.

52. Let F be a field and $D : F[X] \to F[X]$ the map given by

$$D(a_0 + a_1 X + \cdots + a_n X) = a_1 + 2a_2 X + \cdots n a_n X^{n-1}.$$

   (a) (a)] Show that $D(fg) = D(f)g + fD(g)$.
   (b) Suppose that $f \in F[X]$ is irreducible. Show that if $D(f) \neq 0$ then f has no multiple root in any extension field of F.
   (c) Show that if F has characteristic 0 and $f \in F[X]$ is irreducible then f has no repeated roots.

53. Let $\mathbb{F}$ be a field and define $D : \mathbb{F}[x] \to \mathbb{F}[x]$ by

$$D(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1},$$

where $m = \underbrace{1 + \cdots + 1}_{m} \in \mathbb{F}$.

   (a) Verify that $D(fg) = D(f)g + fD(g)$, for all $f, g \in \mathbb{F}[x]$.

(b) An element α is called a double root of f if $(x-\alpha)^2$ divides f. Prove that α is a double root of f if and only if $f(\alpha)=0$ and $(Df)(\alpha)=0$.

54. Let $E=\mathbb{Q}(\alpha)$, where $\alpha^3-\alpha^2+\alpha+2=0$. Express $(\alpha^2+\alpha+1)(\alpha^2-\alpha)$ and $(\alpha-1)^{-1}$ in the form $a\alpha^2+b\alpha+c$ with $a,b,c\in\mathbb{Q}$.

55. Let $F\subseteq K$ be a field extension and let $a\in K$. Under which condition do we call a algebraic over F? Under which condition do we call a transcendental over F?

56. Let $F\subseteq K$ be a field extension and let $a\in K$. Assume that a is algebraic over F. What is the definition of the irreducible polynomial of a over F?

57. Let $F\subseteq K$ be a field extension and let $a\in K$. Assume that F and K are finite fields. Determine (with proof) whether a is algebraic or transcendental.

58. Let $p\in\mathbb{Z}_{>0}$ be prime, let $n\in\mathbb{Z}_{>0}$ and let $\mathbb{F}$ be a finite field of size p^n.

    (a) Show that the map $\varphi\colon\mathbb{F}\to\mathbb{F}$ given by $\varphi(x)=x^p$ is an isomorphism.

    (b) Show that φ has order n.

    (c) Show that every automorphism of $\mathbb{F}$ is a power of φ.

59. Define what it means to say that an element $a\in\mathbb{E}\supseteq\mathbb{F}$ is algebraic over $\mathbb{F}$.

60. Let F be a field. Define the term *splitting field* of a polynomial $f\in F[x]$.

61. Let F be a field. Let $f\in F[x]$. Show that a splitting field of f exists.

62. Show that $\mathbb{Q}(5^{1/3})$ is not the splitting field of any polynomial over $\mathbb{Q}$.

63. Let E and F be fields with F a subfield of E. What does it mean to say that $a\in E$ is algebraic over F?

64. Define what it means to say that an element $a\in\mathbb{C}$ is algebraic over $\mathbb{Q}$.

65. Show that the set of real numbers that are algebraic over $\mathbb{Q}$ is a subfield of $\mathbb{R}$.

66. Suppose that $a\in\mathbb{C}$ is algebraic over $\mathbb{Q}$ and let $n=\deg(a,\mathbb{Q})$. Show that there are exactly n injective field homomorphisms $\mathbb{Q}(q)\to\mathbb{C}$.

67. Let E and F be fields with F a subfield of E. Let $a\in E$ be algebraic over F. Denote by $F(a)$ the smallest subfield of E that contains F and q. Prove that $[F(a):F]=\deg(a,F)$.

68. Let E and F be fields with F a subfield of E. Let $a\in E$ be algebraic over F. Denote by $F[a]$ the smallest subring of E that contains both F and a. Prove that $F(a)=F[a]$.

69. Show that X^2-3 and X^2-2X-2 have the same splitting field K over $\mathbb{Q}$.

70. Let K be the splitting field of X^2-3 over $\mathbb{Q}$. Find $[K:\mathbb{Q}]$.