2.1 Proof of the correspondence theorem

Theorem 2.1. *(Correspondence theorem) Let R be a ring. Let M be an R-module and let N be an R-submodule of M. Let*

$$\mathcal{S}_{[N,M]} = \{R\text{-modules } P \text{ with } N \subseteq P \subseteq M\} \qquad \text{partially ordered by inclusion.}$$

Then the map

$$\varphi\colon \mathcal{S}_{[N,M]} \to \mathcal{S}_{[0,M/N]} \qquad \text{given by} \qquad \varphi(P) = \{p + N \mid p \in P\},$$

is an isomorphism of posets with inverse given by

$$\psi\colon \mathcal{S}_0^{M/N} \to \mathcal{S}_{[N,M]} \qquad \text{given by} \qquad \psi(\Gamma) = \{m \in M \mid m + N \in \Gamma\}.$$

Proof.
To show: (a) φ is a morphism of posets.
(b) ψ is a morphism of posets.
(c) $\psi \circ \varphi = \text{id}$.
(d) $\varphi \circ \psi = \text{id}$.

(a) To show: If $P, Q \in \mathcal{S}_{[N,M]}$ and $P \subseteq Q$ then $\varphi(P) \subseteq \varphi(Q)$.
Assume $P, Q \in \mathcal{S}_{[N,M]}$ and $P \subseteq Q$.
To show: $\varphi(P) \subseteq \varphi(Q)$.
To show: If $x \in \varphi(P)$ then $x \in \varphi(Q)$.
Assume $x \in \varphi(P)$.
Then $x \in \{p + N \mid p \in P\}$.
So there exists $p \in P$ such that $x = p + N$.
Since $P \subseteq Q$ then $p \in Q$.
So $x = p + N$ is an element of $\{q + N \mid q \in Q\} = \varphi(Q)$.
So $\varphi(P) \subseteq \varphi(Q)$.
So φ is a morphism of posets.

(b) To show: If $\Gamma, \Delta \in \mathcal{S}_{[0,M/N]}$ and $\Gamma \subseteq \Delta$ then $\psi(\Gamma) \subseteq \psi(\Delta)$.
Assume $\Gamma, \Delta \in \mathcal{S}_{[0,M/N]}$ and $\Gamma \subseteq \Delta$.
To show: $\psi(\Gamma) \subseteq \psi(\Delta)$.
To show: If $y \in \psi(\Gamma)$ then $y \in \psi(\Delta)$.
Assume $y \in \psi(\Gamma)$.
Then $y \in \{m \in M \mid m + N \in \Gamma\}$.
So $y + N \in \Gamma$.
Since $\Gamma \subseteq \Delta$ then $y \in \Delta$.
So $y \in \{m \in M \mid m + N \in \Delta\} = \psi(\Delta)$.
So $\psi(\Gamma) \subseteq \psi(\Delta)$.
So ψ is a morphism of posets.

(c) To show: $\psi \circ \psi = \text{id}$.
To show:: If $P \in \mathcal{S}_{[N,M]}$ then $\psi(\varphi(P)) = \text{id}(P)$.
Assume $P \in \mathcal{S}_{[N,M]}$.
To show $\psi(\varphi(P)) = P$.
To show: (ca) $\psi(\varphi(P)) \subseteq P$.
(cb) $P \subseteq \psi(\varphi(P))$.

(ca) To show: if $y \in \psi(\varphi(P))$ then $y \in P$.
Assume $y \in \psi(\varphi(P))$.
Then $y \in \{m \in M \mid m + N \in \varphi(P)\}$.
So $y + N \in \varphi(P)$.
So $y + N \in \{p + N \mid p \in P\}$.
So there exists $p \in P$ such that $y + N = p + N$.
So $y \in p + N$.
To show: $y \in P$.
There exists $n \in N$ such that $y = p + n$.
Since $N \subseteq P$ then $y = p + n \in P$.
So $y \in P$.
So $\psi(\varphi(P)) \subseteq P$.

(cb) To show: $P \subseteq \psi(\varphi(P))$.
To show: If $p \in P$ then $p \in \psi(\varphi(P))$.
Assume $p \in P$.
To show: $p \in \psi(\varphi(P))$.
To show: $p \in \{m \in M \mid m + N \in \varphi(P)\}$.
To show: $p + N \in \varphi(P)$.
Since $p \in N$ and $\varphi(P) = \{y + N \mid y \in P\}$ then $p + N \in \varphi(P)$.
So $p \in \{m \in M \mid m + N \in \varphi(P)\}$.
So $p \in \psi(\varphi(P))$.
So $P \subseteq \psi(\varphi(P))$.

So $\psi(\varphi(P)) = P$.

(d) To show: $\varphi \circ \psi = \mathrm{id}$.
To show: If $\Gamma \in \mathcal{S}_{[0,M/N]}$ then $\varphi(\psi(\Gamma)) = \mathrm{id}(\Gamma)$.
Assume $\Gamma \in \mathcal{S}_{[0,M/N]}$.
To show: $\varphi(\psi(\Gamma)) = \Gamma$.
To show: (da) $\varphi(\psi(\Gamma)) \subseteq \Gamma$.
(db) $\Gamma \subseteq \varphi(\psi(\Gamma))$.

(da) To show: if $y \in \varphi(\psi(\Gamma))$ then $y \in \Gamma$.
Assume $y \in \varphi(\psi(\Gamma))$.
Then $y \in \{p + N \mid p \in \psi(\Gamma)\}$.
So there exists $p \in \psi(\Gamma)$ such that $y = p + N$.
So there exists $p \in \{m \in M \mid m + N \in \Gamma\}$ such that $y = p + N$.
So $p + N \in \Gamma$ giving $y \in \Gamma$.
So $\varphi(\psi(\Gamma)) \subseteq \Gamma$.

(db) To show: $\Gamma \subseteq \varphi(\psi(\Gamma))$.
To show: if $X \in \Gamma$ then $X \in \varphi(\psi(\Gamma))$.
Assume $X \in \Gamma$.
Then there exists $m \in M$ such that $X = m + N$ and $m + N \in \Gamma$.
To show: $X \in \varphi(\psi(\Gamma))$.
To show: $X \in \{p + N \mid p \in \psi(\Gamma)\}$.
To show: There exists $p \in \psi(\Gamma)$ such that $X = p + N$.
Let $p = m$.
Then $p \in \{m \in M \mid m + N \in \Gamma\} = \psi(\Gamma)$ and $X = m + N = p + N$.
So $X \in \{p + N \mid p \in \psi(\Gamma)\}$.
So $X \in \varphi(\psi(\Gamma))$.

So $\Gamma \subseteq \varphi(\psi(\Gamma))$.

So $\varphi(\psi(\Gamma)) = \Gamma$.
So $\varphi \circ \psi = \mathrm{id}$.

$\square$