2.21 Proof that finitely generated modules over a PID are direct sums of cyclics

Proposition 2.26. *Let $\mathbb{A}$ be a PID and let M be an $\mathbb{A}$-module given by generators*

$$\textit{generators}\qquad m_1,\ldots,m_s\in M\qquad\textit{and relations}\qquad \begin{array}{c} a_{11}m_1+\cdots+a_{1s}m_s=0,\\ \vdots\\ a_{t1}m_1+\cdots+a_{ts}m_s=0,\end{array}$$

Let $P\in GL_t(\mathbb{A})$, $Q\in GL_s(\mathbb{A})$, $k=\min(s,t)$ and $d_1,\ldots,d_k\in\mathbb{A}$ such that

$$A=PDQ,\qquad\textit{where}\qquad D=\mathrm{diag}(d_1,\ldots,d_k).$$

Then M is presented by

$$\textit{generators}\quad b_1,\ldots,b_s\quad\textit{and relations}\qquad d_1b_1=0,\ldots,d_kb_k=0.$$

Proof. For $i\in\{1,\ldots,s\}$ let

$$b_i=Q_{i1}m_1+\cdots+Q_{is}m_s,\qquad\text{so that}\qquad m_j=(Q^{-1})_{j1}b_1+\cdots+(Q^{-1})_{js}b_s,$$

for $j\in\{1,\ldots,s\}$. Thus generators (m) can be written in terms of generators (b) and vice versa. Since

$$\sum_j a_{ij}m_j=\sum_{j,k}a_{ij}Q^{-1}_{jk}b_k=\sum_k P_{ik}d_kb_k=0$$

then the relations (m) can be derived from the relations (b). Since

$$d_kb_k=\sum_{i,j,l}(P^{-1})_{kj}a_{jl}(Q^{-1})_{lk}b_k=\sum_{i,j,l}(P^{-1})_{kj}a_{jl}m_l=0,$$

then the relations (b) can be derived from the relations (m). □

Theorem 2.27. *Let $\mathbb{A}$ be a PID and let M be a finitely generated $\mathbb{A}$ module. Then there exist $k,\ell\in\mathbb{Z}_{\ge0}$ and $d_1,\ldots,d_k\in\mathbb{A}$ such that*

$$M\cong\frac{\mathbb{A}}{d_1\mathbb{A}}\oplus\cdots\oplus\frac{\mathbb{A}}{d_k\mathbb{A}}\oplus\mathbb{A}^{\oplus\ell}$$

Proof. Since M is finitely generated there exist $s\in\mathbb{Z}_{>0}$ and $m_1,\ldots,m_s\in M$ such that

$$M=\mathbb{A}\text{-span}\{m_1,\ldots,m_s\},\qquad\text{Define}\quad\begin{array}{ccc}\mathbb{A}^{\oplus s}&\xrightarrow{\Phi}&M\\ e_i&\longmapsto&m_i\end{array}\quad\text{and let}\qquad K=\ker(\Phi).$$

Since $\mathbb{A}$ satisfies ACC and $\mathbb{A}^{\oplus s}$ is a finitely generated $\mathbb{A}$-module then

the $\mathbb{A}$-submodule K is finitely generated.

So there exist $t\in\mathbb{Z}_{>0}$ and

$$a_1=(a_{11},\ldots,a_{1s}),\quad\ldots\quad a_t=(a_{t1},\ldots,a_{ts})\qquad\text{in }\mathbb{A}^{\oplus s}\qquad\text{such that}\qquad K=\mathbb{A}\text{-span}\{a_1,\ldots,a_t\}.$$

Since

$$M\cong\frac{\mathbb{A}^{\oplus s}}{K}$$

then M is presented by

$$\text{generators}\qquad m_1,\ldots,m_s\in M\qquad\text{and relations}\qquad \begin{array}{c} a_{11}m_1+\cdots+a_{1s}m_s=0,\\ \vdots\\ a_{t1}m_1+\cdots+a_{ts}m_s=0,\end{array}$$

Then use the previous proposition to produce the isomorphism $M\cong\frac{\mathbb{A}}{d_1\mathbb{A}}\oplus\cdots\oplus\frac{\mathbb{A}}{d_k\mathbb{A}}\oplus\mathbb{A}^{\oplus\ell}$. □