1.17 Lecture 15: Fields of fractions and polynomial rings

1.17.1 Fields of fractions

Definition. Let R be an integral domain.

- A **fraction** is an expression $\dfrac{a}{b}$ with $a \in R$, $b \in R$ and $b \neq 0$.

Proposition 1.77. *Let R be an integral domain. Let $F_R = \left\{\dfrac{a}{b} \mid a, b \in R, b \neq 0\right\}$ be the set of fractions. Define two fractions $\frac{a}{b}$, $\frac{c}{d}$ to be equal if $ad = bc$, i.e.*

$$\frac{a}{b} = \frac{c}{d} \quad \text{if } ad = bc.$$

Then equality of fractions is an equivalence relation on F_R.

Proposition 1.78. *Let R be an integral domain. Let $F_R = \left\{\dfrac{a}{b} \mid a, b \in R, b \neq 0\right\}$ be its set of fractions with equality of fractions be as defined in Proposition 2.30. Then the operations $+\colon F_R \times F_R \to F$ and $\times\colon F_R \times F_R \to F_R$ given by*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \text{are well defined.}$$

Theorem 1.79. *Let R be an integral domain and let $F_R = \left\{\dfrac{a}{b} \mid a \in R, b \in R - \{0\}\right\}$ be the set of fractions with equality of fractions be as defined in Proposition 2.30 and let operations $+\colon F_R \times F_R \to F_R$ and $\times\colon F_R \times F_R \to F_R$ be as given in Proposition 2.31. Then F_R is a field.*

Definition. Let R be an integral domain.

- The **field of fractions** of R is the set $F_R = \{\dfrac{m}{n} \mid m, n \in R, n \neq 0\}$ of fractions with **equality of fractions** defined by
$$\frac{m}{n} = \frac{p}{q} \quad \text{if } mq = np$$
and operations of **addition** $+\colon F_R \times F_R \to F_R$ and **multiplication** $\times\colon F_R \times F_R \to F_R$ defined by
$$\frac{m}{n} + \frac{p}{q} = \frac{mq+np}{pq} \qquad \text{and} \qquad \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}.$$

HW: Give an example of an integral domain R and its field of fractions.

Proposition 1.80. *Let R be an integral domain with identity 1 and let F_R be its field of fractions. Then the map $\varphi\colon R \to F_R$ given by*

$$\begin{array}{rccc} \varphi\colon & R & \to & F_R \\ & r & \mapsto & \frac{r}{1} \end{array}$$

is an injective ring homomorphism.

1.17.2 Polynomial Rings

Definition. Let $\mathbb{A}$ be a commutative ring and for $i \in \mathbb{Z}_{>0}$ let x^i be a formal symbol.

- A **polynomial with coefficients in** $\mathbb{A}$ is an expression of the form
$$a(x) = a_0 + r_1 x + a_2 x^2 + \cdots$$
such that
  (a) if $i \in \mathbb{Z}_{\geq 0}$ then $a_i \in \mathbb{A}$,
  (b) There exists $N \in \mathbb{Z}_{>0}$ such that if $i \in \mathbb{Z}_{>N}$ then $a_i = 0$.

- Polynomials $f(x) = r_0 + r_1 x + r_2 x^2 + \cdots$ and $g(x) = s_0 + s_1 x + s_2 x^2 + \cdots$ with coefficients in R are
$$\textbf{equal} \text{ if } \quad r_i = s_i \text{ for } i \in \mathbb{Z}_{\geq 0}.$$

- The **zero polynomial** is the polynomial $0 = 0 + 0x + 0x^2 + \cdots$.

- The **degree** $\deg\big(f(x)\big)$ of a polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ with coefficients in $\mathbb{A}$ is
$$\text{the smallest } N \in \mathbb{Z}_{\geq 0} \text{ such that } \quad a_N \neq 0 \text{ and } a_k = 0 \text{ for } k \in \mathbb{Z}_{>N}.$$
If $f(x) = 0 + 0x + 0x^2 + \cdots$ then define $\deg\big(f(x)\big) = 0$.

- Let $\mathbb{A}$ be a commutative ring. The **ring of polynomials with coefficients in** $\mathbb{A}$ is the set $\mathbb{A}[x]$ of polynomials with coefficients in $\mathbb{A}$ with the operations of addition and multiplication defined as follows:
If $f(x), g(x) \in \mathbb{A}[x]$ with
$$f(x) = r_0 + r_1 x + r_2 x^2 + \cdots \quad \text{and} \quad g(x) = s_0 + s_1 x + s_2 x^2 + \cdots,$$
then
$$f(x) + g(x) = (r_0 + s_0) + (r_1 + s_1)x + (r_2 + s_2)x^2 + \cdots, \quad \text{and}$$
$$f(x)g(x) = c_0 + c_1 x + c_2 x^2 + \cdots, \quad \text{where} \quad c_k = \sum_{i+j=k} r_i s_j.$$

Proposition 1.81.

(a) Let R, S be commutative rings and let $\varphi\colon R \to S$ be a ring homomorphism. Then the function
$$\begin{array}{ccc} \psi\colon R[x] & \longrightarrow & S[x] \\ r_0 + r_1 x + r_2 x^2 + \cdots & \longmapsto & \varphi(r_0) + \varphi(r_1)x + \varphi(r_2)x^2 + \cdots \end{array}$$
is a ring homomorphism.

(b) Let $R \subseteq S$ be an inclusion of commutative rings and let $\alpha \in R$. Then the evaluation homomorphism
$$\begin{array}{cccc} \mathrm{ev}_{\alpha,S}\colon & R[x] & \to & S \\ & r_0 + r_1 x + \cdots r_d x^d & \mapsto & r_0 + r_1\alpha + \cdots + r_d\alpha^d \end{array} \quad \textit{is a ring homomorphism.}$$

1.17.3 Transport of ring properties to the polynomial ring

Theorem 1.82.

(a) If $\mathbb{A}$ is a commutativering then $\mathbb{A}[x]$ is a commutativering.

(b) If $\mathbb{A}$ is an integral domain then $\mathbb{A}[x]$ is an integral domain.

(c) If $\mathbb{F}$ is a field then $\mathbb{F}[x]$ is a Euclidean domain with size function

$$\begin{array}{rccc} \deg\colon & \mathbb{F}[x]-\{0\} & \to & \mathbb{Z}_{\geq 0} \\ & f(x) & \mapsto & \deg(f(x)). \end{array}$$

(d) If $\mathbb{A}$ satisfies ACC then $\mathbb{A}[x]$ satisfies ACC.

(e) If $\mathbb{A}$ is a UFD then $\mathbb{A}[x]$ is a UFD.

HW: Show that $\mathbb{Z}$ is a PID and $\mathbb{Z}[x]$ is not a PID.