12.03.2024 ①
Algebra Lect 8
A. Ram

Reduction to diagonal

In $M_{3\times 3}(\mathbb{Q}[x])$ let

$$L_1(c)=\begin{pmatrix}1&0&0\\c&1&0\\0&0&1\end{pmatrix}\quad U_1(c)=\begin{pmatrix}1&c&0\\0&1&0\\0&0&1\end{pmatrix},\quad Y_1(c)=\begin{pmatrix}c&1&0\\1&0&0\\0&0&1\end{pmatrix}$$

$$L_2(c)=\begin{pmatrix}1&0&0\\0&1&0\\0&c&1\end{pmatrix}\quad U_2(c)=\begin{pmatrix}1&0&0\\0&1&c\\0&0&1\end{pmatrix}\quad Y_2(c)=\begin{pmatrix}1&0&0\\0&c&1\\0&1&0\end{pmatrix}.$$

These matrices have determinant ± 1 and are in $GL_3(\mathbb{Q}[x])$. Then

$$\begin{pmatrix}2-x&1&-3\\0&-1-x&9\\0&-1&5-x\end{pmatrix}=\begin{pmatrix}1&2-x&-3\\-1-x&0&9\\-1&0&5-x\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&0&-3\\-1-x&(1+x)(2-x)&9\\-1&2-x&5-x\end{pmatrix}\begin{pmatrix}1&2-x&0\\0&1&0\\0&0&1\end{pmatrix}Y_1(0)$$

$$=\begin{pmatrix}1&-3&0\\-1-x&9&(1+x)(2-x)\\-1&5-x&2-x\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}U_1(2-x)Y_1(0)$$

$$=\begin{pmatrix}1&0&0\\-1-x&6-3x&(1+x)(2-x)\\-1&2-x&2-x\end{pmatrix}\begin{pmatrix}1&-3&0\\0&1&0\\0&0&1\end{pmatrix}Y_2(0)U_1(2-x)Y_1(0)$$

$$=\begin{pmatrix}1&0&0\\-1-x&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&6-3x&(1+x)(2-x)\\-1&2-x&2-x\end{pmatrix}U_1(-3)Y_2(0)U_1(2-x)Y_1(0)$$

$$= L_1(-1-x)\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\-1&2-x&2-x\\0&3(2-x)&(1+x)(2-x)\end{pmatrix}$$

$$\cdot u_1(-3)y_2(0)u_1(2-x)y_1(0)$$

$$= L_1(-1-x)y_2(0)\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2-x&2-x\\0&3(2-x)&(1+x)(2-x)\end{pmatrix}$$

$$\cdot u_1(-3)y_2(0)u_1(2-x)y_1(0)$$

$$= L_1(-1-x)y_2(0)L_1(-1)\begin{pmatrix}1&0&0\\0&2-x&0\\0&3(2-x)&(-2+x)(2-x)\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}$$

$$\cdot u_1(-3)y_2(0)u_1(2-x)y_1(0)$$

$$= L_1(-1-x)y_2(0)L_1(-1)\begin{pmatrix}1&0&0\\0&1&0\\0&3&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2-x&0\\0&0&-(2-x)^2\end{pmatrix}$$

$$\cdot u_2(1)u_1(-3)u_1(2-x)y_1(0)$$

Let $P = L_1(-1-x)y_2(0)L_1(-1)L_2(3)$ and

$Q = u_2(1)u_1(-3)u_1(2-x)y_1(0)$.

Then $P, Q \in GL_3(\mathbb{Q}[x])$ and

$$\begin{pmatrix}2-x&1&-3\\0&-1-x&9\\0&-1&5-x\end{pmatrix} = P\begin{pmatrix}1&0&0\\0&2-x&0\\0&0&-(2-x)^2\end{pmatrix}Q.$$

Theorem Let $s, t \in \mathbb{Z}_{>0}$, $k = \min\{s, t\}$.

(a) Let $A \in M_{t\times s}(\mathbb{Z})$. Then there exist

$P \in GL_t(\mathbb{Z})$ and $Q \in GL_s(\mathbb{Z})$ and

$d_1, \ldots, d_k$ with $d_1\mathbb{Z} \supseteq d_2\mathbb{Z} \supseteq \cdots \supseteq d_k\mathbb{Z}$

such that

$$A = PDQ \quad \text{with } D = \mathrm{diag}(d_1, \ldots, d_k)$$

(b) Let $A \in M_{t\times s}(\mathbb{Q}[x])$. Then there exist

$P \in GL_t(\mathbb{Q}[x])$, $Q \in GL_s(\mathbb{Q}[x])$ and

$d_1, \ldots, d_k \in \mathbb{Q}[x]$ with $d_1\mathbb{Q}[x] \supseteq \cdots \supseteq d_k\mathbb{Q}[x]$

such that

$$A = PDQ \quad \text{with } D = \mathrm{diag}(d_1, \ldots, d_k)$$

(c) Let $\mathbb{A}$ be a PID and $A \in M_{t\times s}(\mathbb{A})$.
Then there exist

$P \in GL_t(\mathbb{A})$, $Q \in GL_s(\mathbb{A})$ and

$d_1, \ldots, d_k \in \mathbb{A}$ with $d_1\mathbb{A} \supseteq \cdots \supseteq d_k\mathbb{A}$

such that

$$A = PDQ \quad \text{with } D = \mathrm{diag}(d_1, \ldots, d_k).$$

A PID, or principal ideal domain, is a commutative ring $\mathbb{A}$ that satisfies

(a) (Cancellation Law) If $a, b, c \in \mathbb{A}$ and $c \neq 0$ and $ac = bc$ then $a = b$.

(b) (Principal ideals) If J is an ideal of $\mathbb{A}$ then there exists $m \in \mathbb{A}$ such that
$$J = m\mathbb{A} = \mathbb{A}\text{-span}\{m\}.$$

An ideal of $\mathbb{A}$ is an $\mathbb{A}$-submodule of $\mathbb{A}$.

The group of units of $\mathbb{A}$ is
$$\mathbb{A}^\times = \{a \in \mathbb{A} \mid \text{there exists } b \in \mathbb{A} \text{ with } ba = ab = 1\}.$$

The group of units of $M_{t\times t}(\mathbb{A})$ is
$$GL_t(\mathbb{A}) = \left\{P \in M_{t\times t}(\mathbb{A}) \;\middle|\; \begin{matrix}\text{there exists } S \in M_{t\times t}(\mathbb{A}) \\ \text{with } PS = 1 \text{ and } SP = 1\end{matrix}\right\}$$

Let $s, t \in \mathbb{Z}_{>0}$ and $k = \min(t, s)$.

$E_{ij} \in M_{t\times s}(\mathbb{A})$ is the $s \times t$ matrix with

1, in the (i,j) entry and

0, elsewhere.

Let $d_1, \ldots, d_k \in \mathbb{A}$.

$$\mathrm{diag}(d_1, \ldots, d_k) = d_1 E_{11} + \cdots + d_k E_{kk}$$

$$= \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ 0 & & \ddots & \\ & & & d_k \end{pmatrix} \in M_{k \times s}(\mathbb{A}).$$

The group $\mathbb{A}^\times$ acts on the set $\mathbb{A}$.

The $\mathbb{A}^\times$-orbits are the sets

$$d\mathbb{A}^\times = \{ dc \mid c \in \mathbb{A}^\times \} \quad \text{and}$$

$$\mathbb{A}/\mathbb{A}^\times = \{ d\mathbb{A}^\times \mid d \in \mathbb{A} \}$$

is the <u>set of $\mathbb{A}^\times$-orbits</u>.

<u>Proposition</u> Let $\mathbb{A}$ be a PID

The map

$$\{\text{ideals of } \mathbb{A}\} \longleftrightarrow \mathbb{A}/\mathbb{A}^\times$$
$$d\mathbb{A} \longleftarrow d\mathbb{A}^\times$$

is a bijection.

<u>Favorite examples:</u>

(1) $\mathbb{A} = \mathbb{Z}$

(2) $\mathbb{A} = \mathbb{F}[x]$, where $\mathbb{F}$ is a field

(3) $\mathbb{A} = \mathbb{F}$, where $\mathbb{F}$ is a field.