14.03.2024

Algebra Lect 9 ①

A. Ram

Generators and Relations

Let $\mathbb{A} = \mathbb{Z}$. Then $\mathbb{Z}$ is a favorite $\mathbb{Z}$-module.

Since

$$\mathbb{Z} = \mathbb{Z}\text{-span}\{1\} \text{ then } \mathbb{Z} \text{ is generated by } 1.$$

$\mathbb{Z}/7\mathbb{Z}$ is another $\mathbb{Z}$-module. Since

$$\mathbb{Z}/7\mathbb{Z} = \mathbb{Z}\text{-span}\{1\} = \{0, 1, 2, 3, 4, 5, 6\}$$

then $\mathbb{Z}/7\mathbb{Z}$ is generated by 1 and

$$7 \cdot 1 = 0 \text{ is a relation in } \mathbb{Z}/7\mathbb{Z}.$$

Let M be an $\mathbb{A}$-module and let $m_1, \ldots, m_k \in M$. The module M is <u>generated</u> by $\{m_1, \ldots, m_k\}$ if

$$M = \mathbb{A}\text{-span}\{m_1, \ldots, m_k\}.$$

A <u>relation</u> is a linear combination $a_1 m_1 + \cdots + a_k m_k$ such that $a_1 m_1 + \cdots + a_k m_k = 0$ in M.

Let $k \in \mathbb{Z}_{>0}$.

The <u>free module of rank k</u> is

$$\mathbb{A}^{\oplus k} = M_{k\times 1}(\mathbb{A}) = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} \;\middle|\; a_1, \ldots, a_k \in \mathbb{A} \right\}$$

The module $\mathbb{A}^{\oplus k}$ is generated by $\{e_1, \ldots, e_k\}$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The module $\mathbb{A}^{\oplus k} = \mathbb{A} \oplus \cdots \oplus \mathbb{A}$ has <u>no relations</u>

Let $d_1, \ldots, d_k \in \mathbb{A}$.

$$\frac{\mathbb{A}}{d_1\mathbb{A}} \oplus \cdots \oplus \frac{\mathbb{A}}{d_k\mathbb{A}} = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} \;\middle|\; a_i \in \frac{\mathbb{A}}{d_i\mathbb{A}} \right\}$$

is generated by $\{e_1, \ldots, e_k\}$ with relations

$$d_1 e_1 = 0, \quad d_2 e_2 = 0, \quad \ldots, \quad d_k e_k = 0.$$

Theorem/Construction

Let $\mathbb{A}$ be a PID.

Let M be an $\mathbb{A}$-module given by generators

$m_1, \dots, m_s \in M$ with relations $a_{11}m_1 + \cdots + a_{1s}m_s = 0,$

$$\vdots$$

$$a_{t1}m_1 + \cdots + a_{ts}m_s = 0.$$

Let $A \in M_{t\times s}(\mathbb{A})$ so that the (i,j) entry of A is a_{ij}. Let $P \in GL_t(\mathbb{A})$ and $Q \in GL_t(\mathbb{A})$ and $d_1, \dots, d_k \in \mathbb{A}$ such that $d_1\mathbb{A} \supseteq d_2\mathbb{A} \supseteq \cdots \supseteq d_k\mathbb{A}$

and $A = PDQ$, where $D = \mathrm{diag}(d_1, \dots, d_k)$.

For $i \in \{1, \dots, s\}$ let

$$b_i = Q_{i1}m_1 + \cdots + Q_{is}m_s.$$

Then M is given by generators

$b_1, \dots, b_s$ and relations
$$d_1 b_1 = 0,$$
$$d_2 b_2 = 0,$$
$$\vdots$$
$$d_k b_k = 0.$$

Thus,

$$M \simeq \frac{\mathbb{A}}{d_1\mathbb{A}} \oplus \cdots \oplus \frac{\mathbb{A}}{d_k\mathbb{A}}.$$

Proof:

To show: (a) Generators (b) can be written in terms of generators (m)

(b) Generators (m) can be written in terms of generators (b).

(c) Relations (b) can be derived from relations (m)

(d) Relations (m) can be derived from relations (b).

(a) and (b) come from

$$b_i = Q_{i1} m_1 + \cdots + Q_{is} m_s \quad \text{for } i \in \{1, \ldots, s\}$$

and

$$m_j = (Q^{-1})_{j1} b_1 + \cdots + (Q^{-1})_{js} b_s \quad \text{for } j \in \{1, \ldots, s\}.$$

(c) Assume

$$a_{11} m_1 + \cdots + a_{1s} m_s = 0,$$
$$\vdots$$
$$a_{t1} m_1 + \cdots + a_{ts} m_s = 0.$$

Then

$$d_i b_i = d_i \sum_{k=1}^{s} Q_{ik} m_k = \sum_{k,l=1}^{s} (P^{-1})_{il} a_{lk} m_k = 0$$

So the relations (b) follow from the relations (m).

(d) Assume $d_1 b_1 = 0, \ldots, d_k b_k = 0$.
Let $d_{k+1} = 0, \ldots, d_s = 0$ so that we can write

$$d_i b_i = 0 \quad \text{for } i \in \{1, \ldots, s\}.$$

Let $i \in \{1, \ldots, t\}$. Then

$$\sum_{j=1}^{s} a_{ij} m_j = \sum_{j,k=1}^{s} a_{ij} (Q^{-1})_{jk} b_k = \sum_{k=1}^{s} P_{ik} d_k b_k = 0.$$

So the relations (m) follow from the relations (b). //.

A <u>simple A-module</u> is an A-module M with no submodules other than 0 and M.

A simple A-module is an A-module M which satisfies:

If $N \subseteq M$ is an A-submodule of M and $N \neq 0$ then $N = M$.