The Krull-Schmidt Theorem

Let
$$M = \mathbb{Z}\text{-span}\{m_1, m_2, m_3, m_4, m_5\}$$
with relations
$$64m_1 + 12m_2 + 36m_3 = 0,$$
$$24m_2 + 48m_4 + 60m_5 = 0,$$
$$120m_3 + 16m_4 + 72m_5 = 0.$$
Make
$$A = \begin{pmatrix} 64 & 12 & 36 & 0 & 0 \\ 0 & 24 & 0 & 48 & 60 \\ 0 & 0 & 120 & 16 & 72 \end{pmatrix}$$
Row reduce to get
$$A = P\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 60 & 0 & 0 \end{pmatrix} Q.$$
Then
$$M \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/60\mathbb{Z} \oplus \mathbb{Z}/0\mathbb{Z} \oplus \mathbb{Z}/0\mathbb{Z}$$
$$\cong \mathbb{Z}/2^2\mathbb{Z} \oplus \mathbb{Z}/2^2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2^2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$$
$$\oplus\, \mathbb{Z} \oplus \mathbb{Z}$$

Krull-Schmidt Theorem

Let $\mathbb{A}$ be a PID.
Let M be an $\mathbb{A}$-module given by a finite number of generators and relations.
Then there exist $k \in \mathbb{Z}_{\geq 0}$,

$p_1, p_2, \ldots, p_k \in \mathbb{A}$ irreducible,

$r_1, r_2, \ldots, r_k \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}_{\geq 0}$

such that

$$M \cong \frac{\mathbb{Z}}{p_1^{r_1}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p_2^{r_2}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k^{r_k}\mathbb{Z}} \oplus \mathbb{Z}^r.$$

Classification of simples and indecomposables

Theorem Let $\mathbb{A}$ be a PID.

$$\begin{array}{ccc} \{\text{simple } \mathbb{A}\text{-modules}\} & \longleftrightarrow & \left\{ p\mathbb{A}^\times \;\middle|\; \begin{array}{l} p \in \mathbb{A}, \\ p \text{ is irreducible} \end{array} \right\} \\ \mathbb{A}/p\mathbb{A} & \longmapsfrom & p \end{array}$$

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{indecomposable} \\ \mathbb{A}\text{-modules} \end{array} \right\} & \longleftrightarrow & \left\{ (p\mathbb{A}^\times, k) \;\middle|\; \begin{array}{l} p \in \mathbb{A}, \\ p \text{ is irreducible} \\ k \in \mathbb{Z}_{>0} \end{array} \right\} \\ \dfrac{\mathbb{A}}{p^k\mathbb{A}} & \longmapsfrom & (p, k) \end{array}$$

Definitions

Let R be a ring and let M be A. Ram
an R-module.

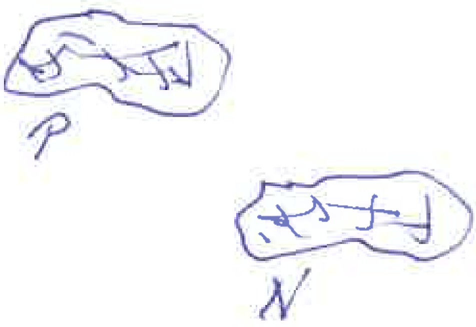
The R-module M is simple if $M \neq 0$ and
if $N \subseteq M$ is an R-submodule of M
and $N \neq 0$ then $N = M$.

The R-module M is indecomposable if
$M \neq 0$ and there do not exist R-submodules
$N \subseteq M$ and $P \subseteq M$ with $N \neq 0$, $P \neq 0$
and $M = N \oplus P$

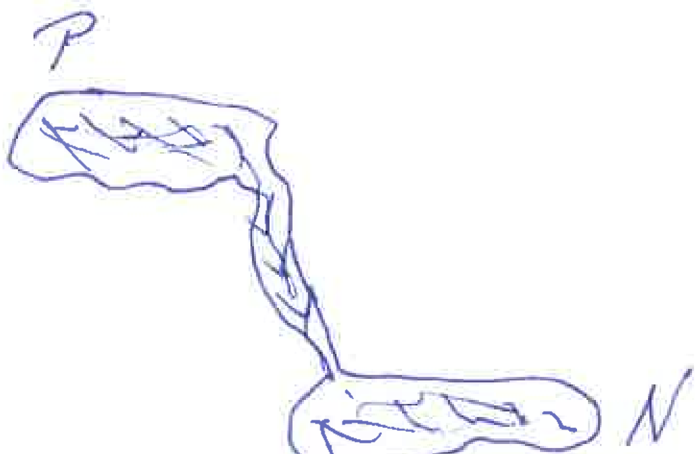
Here $M = N \oplus P$ means

(a) $M = N + P$, where $N + P = \left\{ n + p \;\middle|\; \begin{matrix} n \in N \\ p \in P \end{matrix} \right\}$

(b) $N \cap P = \{0\}$.



$M = P \oplus N$
decomposable



$M \neq P \oplus N$
indecomposable

Chinese decomposition theorem

Let $\mathbb{A}$ be a PID. Let $d \in \mathbb{A}$.

Assume $d = pq$ with $\gcd(p, q) = 1$.

Then $\frac{\mathbb{A}}{d\mathbb{A}} \cong \frac{\mathbb{A}}{p\mathbb{A}} \oplus \frac{\mathbb{A}}{q\mathbb{A}}$.

Proof sketch Let $r, s \in \mathbb{A}$ with $pr + qs = 1$.

$$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & pq \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -qs & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ qs & pq \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -qs & 1 \end{pmatrix}\begin{pmatrix} pr+qs & 0 \\ qs & pq \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -qs & 1 \end{pmatrix}\begin{pmatrix} p & q \\ 0 & q \end{pmatrix}\begin{pmatrix} r & -q \\ s & p \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -qs & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\begin{pmatrix} r & -q \\ s & p \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -qs & 1-qs \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\begin{pmatrix} r & -q \\ s & p \end{pmatrix} = PDQ.$$

$\frac{\mathbb{A}}{d\mathbb{A}}$ is given by generators m_1, m_2

and relations $1 \cdot m_1 = 0$ and $d \cdot m_2 = 0$.

Let

$b_1 = rm_1 - qm_2$ and $b_2 = sm_1 + pm_2$

Then

$$pb_1 + qb_2 = prm_1 - pqm_2 + qsm_1 + pqm_2 = m_1$$

and

$$-sb_1 + rb_2 = -srm_1 + sqm_2 + rsm_1 + prm_2 = m_2$$

and so m_1 and m_2 can be recovered from b_1 and b_2. Then

$$pb_1 = prm_1 - pqm_2 = pr\cdot 0 - dm_2 = 0 - 0 = 0$$

and $qb_2 = qsm_1 + qpm_2 = qs\cdot 0 + dm_2 = 0 + 0 = 0$.

so that the relations

$$pb_1 = 0 \quad \text{and} \quad qb_2 = 0$$

are derived from the relations $1\cdot m_1 = 0$ and $dm_2 = 0$.

Assuming $pb_1 = 0$ and $qb_2 = 0$ then

$1\cdot m_1 = pb_1 + qb_2 = 0 + 0$ and

$dm_2 = pq(-sb_1 + rb_2) = -sqpb_1 + rpqb_2 = 0$.

So the relations $1\cdot m_1 = 0$ and $dm_2 = 0$ can be derived from the relations $pb_1 = 0$ and $qb_2 = 0$.

So $$\frac{A}{dA} \cong \frac{A}{pA} \oplus \frac{A}{qA}.$$