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Algebra Lect 11

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Krull-Schmidt theorem

Let $\mathbb{A} = \mathbb{F}[x]$. Let M be an $\mathbb{F}[x]$-module given by a finite number of generators and relations. Then there exist $k \in \mathbb{Z}_{>0}$ and $p_1(x), \ldots, p_k(x) \in \mathbb{F}[x]$ irreducible and $m_1, \ldots, m_k \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}_{\geq 0}$ such that

$$M \cong \frac{\mathbb{F}[x]}{p_1(x)^{m_1}\mathbb{F}[x]} \oplus \cdots \oplus \frac{\mathbb{F}[x]}{p_k(x)^{m_k}\mathbb{F}[x]} \oplus \mathbb{F}[x]^{\oplus r}.$$

The matrix of the action of x

(1) Let $p(x) = x^d - a_{d-1}x^{d-1} - \cdots - a_1 x - a_0$.

Then $\dfrac{\mathbb{F}[x]}{p(x)\mathbb{F}[x]}$ has $\mathbb{F}$-basis $\{x^{d-1}, \ldots, x, 1\}$.

The matrix of the action of x with respect to the $\mathbb{F}$-basis $\{x^{d-1}, \ldots, x, 1\}$ is

$$J_1(p(x)) = \begin{pmatrix} a_{d-1} & 1 & & 0 \\ \vdots & & \ddots & \\ \vdots & 0 & & \ddots \\ a_1 & & & 1 \\ a_0 & 0 & \cdots & 0 \end{pmatrix} \in M_{d\times d}(\mathbb{F}).$$

(2) $\dfrac{\mathbb{F}[x]}{p(x)^m\mathbb{F}[x]}$ has $\mathbb{F}$-basis

$$B = \{x^{d-1}, \ldots, x, 1, p(x)x^{d-1}, \ldots, p(x)x, p(x),$$
$$\ldots, p(x)^{m-1}x^{d-1}, \ldots, p(x)^{m-1}x, p(x)^{m-1}\}$$

The matrix of the action of x with respect to the basis B is

$$J_m(p(x)) = \begin{pmatrix} J_1(p(x)) & & & & 0 \\ E_{11} & J_1(p(x)) & & & \\ & E_{11} & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & E_{11} & J_1(p(x)) \end{pmatrix}$$

where $E_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \in M_{d\times d}(\mathbb{F})$.

For $B \in M_{s\times s}(\mathbb{F})$ and $C \in M_{t\times t}(\mathbb{F})$ define

$$B \oplus C = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right) \in M_{(s+t,\,s+t)}(\mathbb{F}).$$

If

$$M \cong \frac{\mathbb{F}[x]}{p_1(x)^{m_1}\mathbb{F}[x]} \oplus \cdots \oplus \frac{\mathbb{F}[x]}{p_k(x)^{m_k}\mathbb{F}[x]}$$

then there is a basis B of M such that the matrix of the action of x with respect to the basis B is

$$J = \begin{pmatrix} J_{m_1}(p_1(x)) & & & 0 \\ & J_{m_2}(p_2(x)) & & \\ & & \ddots & \\ 0 & & & J_{m_k}(p_k(x)) \end{pmatrix}$$

<u>Construction of an $\mathbb{F}[x]$-module from a matrix</u>

Let $\mathbb{F}$ be a field. Let $n \in \mathbb{Z}_{>0}$.

Let $A \in M_{n\times n}(\mathbb{F})$

Let $V = \mathbb{F}^n = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mid a_i \in \mathbb{F} \right\}$ with the functions

$$\begin{matrix} V\times V \longrightarrow V \\ (v_1, v_2) \longmapsto v_1+v_2 \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbb{F}[x]\times V \longrightarrow V \\ (c_0+\cdots+c_\ell x^\ell, v) \longmapsto (c_0+\cdots+c_\ell A^\ell) v. \end{matrix}$$

Then V is an $\mathbb{F}[x]$-module.

Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Then V is given by generators $\{e_1, \dots, e_n\}$ and relations

$$x e_1 - Te_1 = 0, \quad \dots, \quad x e_n - T e_n = 0.$$

In English: V is the $\mathbb{F}[x]$-module where the action of x on V, with respect to the $\mathbb{F}$-basis $\{e_1, \dots, e_n\}$ has matrix T.

<u>Example</u> $T = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & -1 \\ -3 & 4 & 5 \end{pmatrix}$

then the generators e_1, e_2, e_3 have relations

$$x e_1 - (2e_1 + e_2 - 3e_3) = 0$$

$$x e_2 - (-e_2 + 4 e_3) = 0$$

$$x e_3 - (-e_2 + 5 e_3) = 0.$$

So the matrix of relations is

$$A = \begin{pmatrix} x-2 & -1 & 3 \\ 0 & 1+x & -4 \\ 0 & 1 & x-5 \end{pmatrix}$$

Row reduce A to write

$$A = PDQ \quad \text{with } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x-2 & 0 \\ 0 & 0 & (x-2)^2 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} x-2 & -1 & +3 \\ -2 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

So we get new generators

$$f_1 = (x-2)e_1 - e_2 + 3e_3 = (2e_1 + e_2 - 3e_3) - 2e_1 - e_2 + 3e_3 = 0$$

$$f_2 = -2e_1 - e_3$$

$$f_3 = -e_3.$$

Then the new relations are

$$1 \cdot f_1 = 0, \quad (x-2)f_2 = 0, \quad (x-2)^2 f_3 = 0$$

and

$$V \cong \frac{\mathbb{F}[x]}{\mathbb{F}[x]} \oplus \frac{\mathbb{F}[x]}{(x-2)\mathbb{F}[x]} \oplus \frac{\mathbb{F}[x]}{(x-2)^2\mathbb{F}[x]}.$$

Let $b_1 = f_2 = -2e_1 - e_3$,
$b_2 = f_3 = -e_3$,
$b_3 = (x-2)f_3$.

Then the matrix of the action of x on V in the basis $\{b_1, b_2, b_3\}$ is

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$