Finiteness conditions

Let R be a ring. Let M be an R-module.

$$S_{[0,M]} = \{\text{submodules of } M\}$$

partially ordered by inclusion.

The module <u>M satisfies ACC</u> if M satisfies

If $0 = M_0 \subseteq M_1 \subseteq \cdots$ is an increasing chain in $S_{[0,M]}$ then there exists $k \in \mathbb{Z}_{>0}$ such that if $\ell \in \mathbb{Z}_{>k}$ then $M_k = M_\ell$.

In English: Increasing chains in $S_{[0,M]}$ are finite.

The module <u>M satisfies DCC</u> if decreasing chains in $S_{[0,M]}$ are finite.

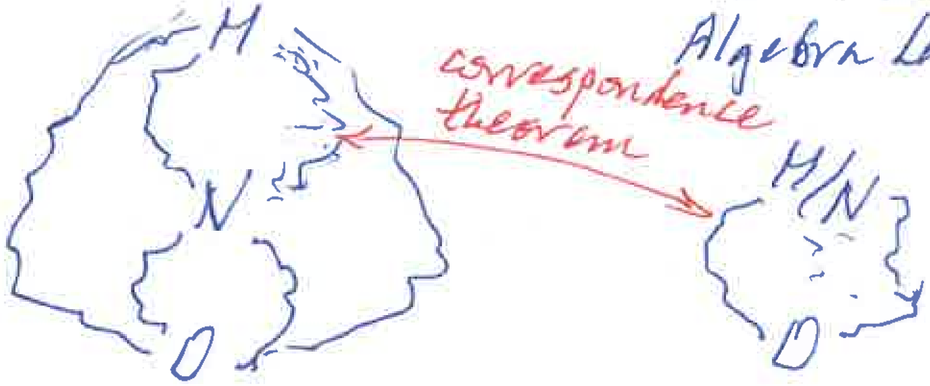
<u>Proposition</u> Let N be a submodule of M

(a) M satisfies ACC if and only if N and M/N satisfy ACC.

(b) M satisfies DCC if and only if N and M/N satisfy DCC

(c) M satisfies both ACC and DCC if and only if N and M/N satisfy both ACC and DCC.

Cartoon:



The module M is <u>finitely generated</u> if there exist $k \in \mathbb{Z}_{>0}$ and $b_1, \ldots, b_k \in M$ such that

$$M = R\text{-span}\{b_1, \ldots, b_k\} = Rb_1 + \cdots + Rb_k$$

<u>Proposition</u> Let N be a submodule of M.

(a) If M is finitely generated then M/N is finitely generated.

(b) M satisfies ACC if and only if every submodule of M is finitely generated

(c) If R satisfies ACC and M is finitely generated then M satisfies ACC.

Jordan-Hölder theorem

The module M is simple if $\mathcal{S}_{[0,M]} = \{0, M\}$.

A finite composition series of M is a chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

such that $n \in \mathbb{Z}_{>0}$ and

if $i \in \{1, \ldots, n\}$ then M_i / M_{i-1} is simple.

Proposition (Jordan-Hölder Theorem)

(a) M has a finite composition series if and only if M satisfies both ACC and DCC.

(b) Any two series

$$0 = M_0 \subseteq \cdots \subseteq M_s = M \quad \text{and}$$

$$0 = N_0 \subseteq \cdots \subseteq N_t = M$$

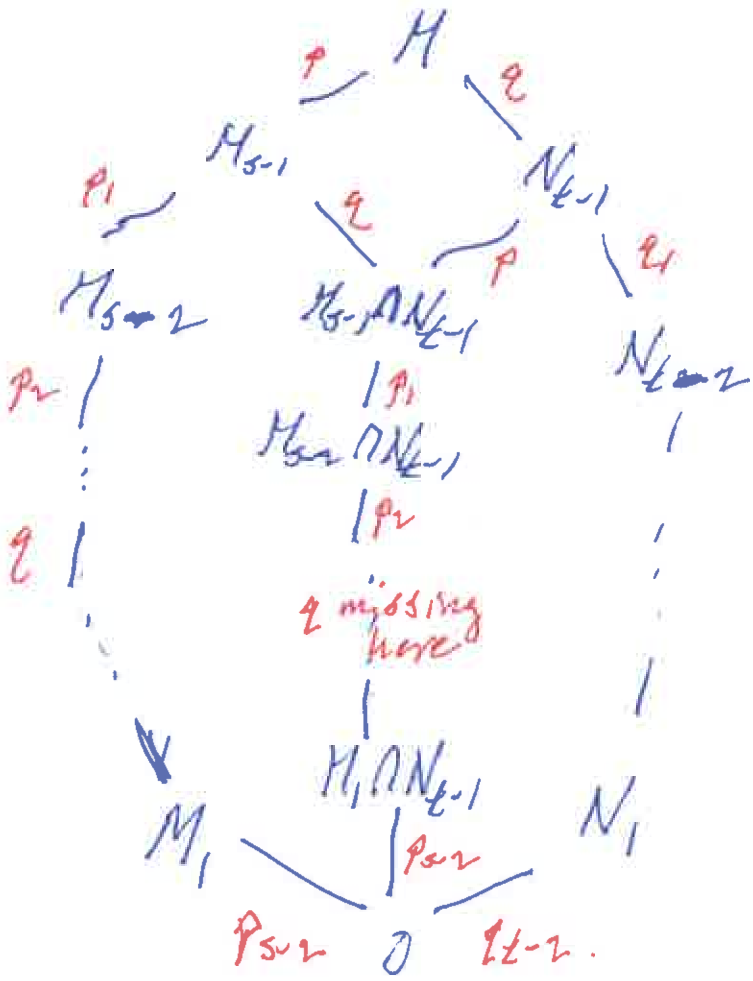
can be refined to have the same length and the same factors.

Proposition (Jordan-Hölder theorem).

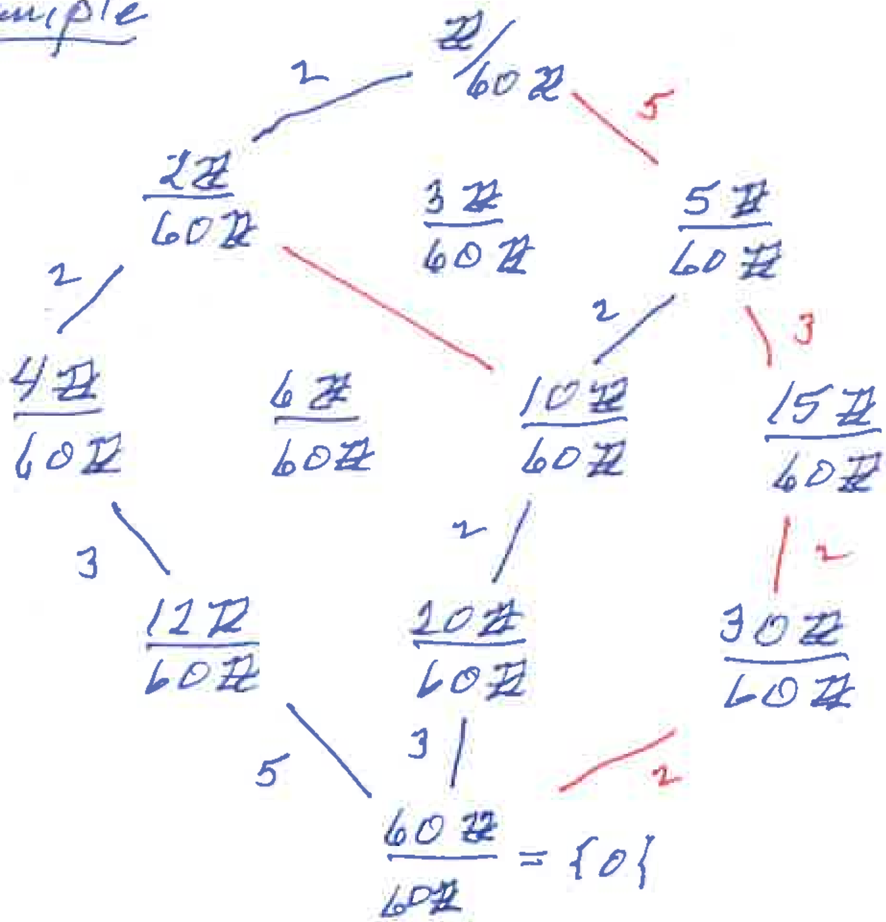
(a) M has a finite composition series
if and only if
M satisfies both ACC and DCC.

(b) Any two series
$$0 = M_0 \subseteq \cdots \subseteq M_k = M \quad \text{and}$$
$$0 = N_0 \subseteq \cdots \subseteq N_t = M$$
can be refined to have the same length
and the same factors.

Greedy refinement



Example

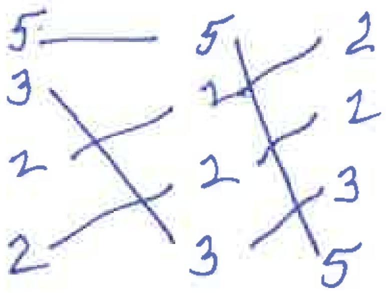


Greedy refinement changes $2\cdot 2\cdot 3\cdot 5$

to $5\cdot 2\cdot 2\cdot 3$

to $5\cdot 3\cdot 2\cdot 2$

which is the sequence of permutations



<u>Symmetric refinement</u> Let

$$0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M \quad \text{and}$$

$$0 \subseteq N_1 \subseteq \cdots \subseteq N_t = M$$

be ascending sequences in $\mathcal{S}_{[0,M]}$.

For $i \in \{1,\ldots,s\}$ and $j \in \{1,\ldots,t\}$ define

$$M_{ij} = (M_{i-1}+N_j)\cap M_i \quad \text{and} \quad N_{ji} = (N_{j-1}+M_i)\cap N_j$$

Let

$$Q_{ij} = \frac{M_{ij}}{M_{i,j-1}} \quad \text{and} \quad F_{ji} = \frac{N_{ji}}{N_{j,i-1}}.$$

Then

$$Q_{ij} \cong F_{ji}.$$

<u>Example</u> Factorizations of $d = 2^2 3^3$.

$$(2^2 3^3\mathbb{Z} \subseteq 2^2 3^2 \mathbb{Z} \subseteq 2^2\mathbb{Z} \subseteq \mathbb{Z}) = (M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3)$$

and

$$(2^2 3^3\mathbb{Z} \subseteq 3^3\mathbb{Z} \subseteq \mathbb{Z}) = (N_0 \subseteq N_1 \subseteq N_2)$$

Then

$$\begin{pmatrix} 2^2 3^3\mathbb{Z} \overset{1}{\subseteq} 2^2 3^3\mathbb{Z} \overset{3}{\subseteq} 2^2 3^2\mathbb{Z} \\ = 2^2 3^2\mathbb{Z} \overset{1}{\subseteq} 2^2 3^2\mathbb{Z} \overset{3^2}{\subseteq} 2^2\mathbb{Z} \\ = 2^2\mathbb{Z} \overset{2^2}{\subseteq} \mathbb{Z} \overset{1}{\subseteq} \mathbb{Z} \end{pmatrix} = \begin{pmatrix} M_{11} \overset{Q_{12}}{\subseteq} M_{12} \overset{Q_{13}}{\subseteq} M_{13} \\ = M_{21} \overset{Q_{22}}{\subseteq} M_{22} \overset{Q_{23}}{\subseteq} M_{23} \\ = M_{31} \overset{Q_{32}}{\subseteq} M_{32} \overset{Q_{33}}{\subseteq} M_{33} \end{pmatrix}$$

and

$$\begin{pmatrix} 2^2 3^3 \mathbb{Z} \overset{1}{\subseteq} 2^2 3^3 \mathbb{Z} \overset{1}{\subseteq} 2^2 3^3 \mathbb{Z} \overset{2^2}{\subseteq} 3^3 \mathbb{Z} \\ = 3^3 \mathbb{Z} \overset{3}{\subseteq} 3^2 \mathbb{Z} \overset{3^2}{\subseteq} \mathbb{Z} \overset{1}{\subseteq} \mathbb{Z} \end{pmatrix}$$

$$= \begin{pmatrix} N_{11} \overset{F_{12}}{\subseteq} N_{12} \overset{F_{13}}{\subseteq} N_{13} \overset{F_{14}}{\subseteq} N_{14} \\ = N_{21} \overset{F_{22}}{\subseteq} N_{22} \overset{F_{23}}{\subseteq} N_{23} \overset{F_{24}}{\subseteq} N_{24} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 3 \\ 1 & 3^2 \\ 2^2 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 & 2^2 \\ 3 & 3^2 & 1 \end{pmatrix}$$