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Algebra Lect. 20 ①
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Jordan Normal Form
$\mathbb{F}[x]$-modules

Theorem (Invariant factor decomposition).
If M is a finitely generated $\mathbb{F}[x]$-module then there exist $d_1, \ldots, d_\ell \in \mathbb{F}[x]$ such that

$$d_1\mathbb{F}[x] \supseteq d_2\mathbb{F}[x] \supseteq \cdots \supseteq d_\ell\mathbb{F}[x] \quad \text{and}$$

$$M \cong \frac{\mathbb{F}[x]}{d_1\mathbb{F}[x]} \oplus \cdots \oplus \frac{\mathbb{F}[x]}{d_\ell\mathbb{F}[x]}$$ (invariant factor decomposition)

The Chinese block decomposition says:

If $\gcd(p,q)=1$ then $\dfrac{\mathbb{F}[x]}{pq\mathbb{F}[x]} \cong \dfrac{\mathbb{F}[x]}{p\mathbb{F}[x]} \oplus \dfrac{\mathbb{F}[x]}{q\mathbb{F}[x]}$.

So, if $p_1, \ldots, p_k \in \mathbb{F}[x]$ are irreducible and distinct and $m_1, \ldots, m_k \in \mathbb{Z}_{>0}$ then

$$\frac{\mathbb{F}[x]}{p_1^{m_1}\cdots p_k^{m_k}\mathbb{F}[x]} \cong \frac{\mathbb{F}[x]}{p_1^{m_1}\mathbb{F}[x]} \oplus \cdots \oplus \frac{\mathbb{F}[x]}{p_k^{m_k}\mathbb{F}[x]}.$$

Theorem A submodule K of $\mathbb{F}[x]^{\oplus r}$ has a basis.

A vector space $\mathbb{F}^n$ and a matrix $A \in M_{n\times n}(\mathbb{F})$ give an $\mathbb{F}[x]$-module, where x acts on $\mathbb{F}^n$ by the matrix A (in the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{F}^n$.)

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<u>Example</u> Let $M = \mathbb{Q}^n$ and let $A \in M_{n\times n}(\mathbb{Q})$.

Assume that, as an $\mathbb{Q}[x]$-module

$$M \simeq \frac{\mathbb{Q}[x]}{(x-1)^3\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x^2+1)^2\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x-1)(x^2+1)^4\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x+2)(x^2+1)^2\mathbb{Q}[x]}$$

<u>Questions:</u>

(a) What is the primary decomposition of M?

(b) What is the invariant factor decomposition of M?

(c) What is the Jordan normal form of A?

(d) Is A diagonalisable?

(e) What is the minimal polynomial of A?

(f) What is the characteristic polynomial of A?

(g) What are the dimensions of the eigenspaces?

(h) What are the dimensions of the generalized eigenspaces?

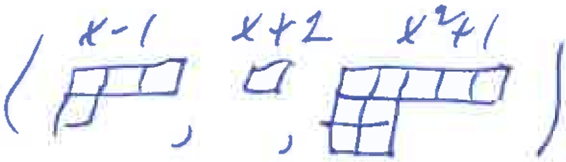
the primary decomposition of M is

$$M \cong \frac{\mathbb{Q}[x]}{(x-1)\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x-1)^3\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x+2)\mathbb{Q}[x]}$$
$$\oplus \frac{\mathbb{Q}[x]}{(x^2+1)^2\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x^2+1)^2\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x^2+1)^4\mathbb{Q}[x]}.$$

The invariant factor decomposition of M is

$$M \cong \frac{\mathbb{Q}[x]}{(x^2+1)^2\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x-1)(x^2+1)^2\mathbb{Q}[x]}$$
$$\oplus \frac{\mathbb{Q}[x]}{(x-1)^3(x+2)(x^2+1)^4\mathbb{Q}[x]}$$

The cartoon for M is



The characteristic polynomial of A is

$$\mathrm{char}(A) = (x-1)^4(x+2)(x^2+1)^8.$$

The minimal polynomial of A is

$$\min_{A,\mathbb{Q}}(x) = (x-1)^3(x+2)(x^2+1)^4.$$

Let

$$J_1(x-1) = (1), \qquad J_3(x-1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Let $J_1(x+2) = (-2)$ and

$$J_2(x^2+1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad J_4(x^2+1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The Jordan normal form of A is

$$J(A) = \begin{pmatrix} J_1(x-1) & 0 & 0 & 0 & 0 & 0 \\ 0 & J_3(x-1) & 0 & 0 & 0 & 0 \\ 0 & 0 & J_1(x+2) & 0 & 0 & 0 \\ 0 & 0 & 0 & J_2(x^2+1) & 0 & 0 \\ 0 & 0 & 0 & 0 & J_2(x^2+1) & 0 \\ 0 & 0 & 0 & 0 & 0 & J_4(x^2+1) \end{pmatrix}$$

The matrix A is not diagonalisable over $\mathbb{Q}$

If we view A as a matrix in $M_{n\times n}(\mathbb{C})$ then

$$\mathbb{C}^{13} \cong \frac{\mathbb{C}[x]}{(x-1)\mathbb{C}[x]} \oplus \frac{\mathbb{C}[x]}{(x-1)^3\mathbb{C}[x]} \oplus \frac{\mathbb{C}[x]}{(x+2)\mathbb{C}[x]}$$

$$\oplus \frac{\mathbb{C}[x]}{(x-i)^2\mathbb{C}[x]} \oplus \frac{\mathbb{C}[x]}{(x-i)^2\mathbb{C}[x]} \oplus \frac{\mathbb{C}[x]}{(x-i)^4\mathbb{C}[x]}$$

$$\oplus \frac{\mathbb{C}[x]}{(x+i)^2\mathbb{C}[x]} \oplus \frac{\mathbb{C}[x]}{(x+i)^2\mathbb{C}[x]} \oplus \frac{\mathbb{C}[x]}{(x+i)^4\mathbb{C}[x]}$$

as $\mathbb{C}[x]$-modules

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The Jordan normal form of A over $\mathbb{C}$ is

$$J_{\mathbb{C}}(A) = \begin{pmatrix}
\boxed{1} & & & & & & & & \\
& \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} & & & & & & 0 & \\
& & \boxed{2} & & & & & & \\
& & & \begin{pmatrix} i & 0 \\ 1 & i \end{pmatrix} & & & & & \\
& & & & \begin{pmatrix} i & 0 \\ 1 & i \end{pmatrix} & & & & \\
& & & & & \begin{pmatrix} i & 0 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1 & i \end{pmatrix} & & & \\
& & 0 & & & & \begin{pmatrix} -i & 0 \\ 1 & -i \end{pmatrix} & & \\
& & & & & & & \begin{pmatrix} -i & 0 \\ 1 & -i \end{pmatrix} & \\
& & & & & & & & \begin{pmatrix} -i & 0 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 1 & -i \end{pmatrix}
\end{pmatrix}$$

The matrix A is not diagonalisable over $\mathbb{C}$.

The dimensions of the eigenspaces are:

Eigenvalue 1: dimension 2

Eigenvalue 2: dimension 1

Eigenvalue i: dimension 3

Eigenvalue $-i$: dimension 3

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The dimensions of the generalized eigenspaces are:

Eigenvalue 1: dimension $1+3=4$,
Eigenvalue 2: dimension 1,
Eigenvalue i: dimension $2+2+4=8$,
Eigenvalue $-i$: dimension $2+2+4=8$.