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Algebra Lect. 21 ①

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Function fields

$\mathbb{K} = \mathbb{C}(t)$ and

automorphisms $\sigma, \tau, \rho \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(t))$ given by

$$\sigma(t) = \frac{1}{t}, \quad \tau(t) = 1-t, \quad \rho(t) = -t.$$

Let

$$u = t^2, \quad v = t^2 - t, \quad w = \frac{(t^2-t+1)^3}{t^2(t-1)^2}$$

Then

$$\rho(u) = \rho(t^2) = (-t)^2 = t^2 = u,$$

$$\tau(v) = \tau(t^2-t) = (1-t)^2 - (1-t) = (1-t)(1-t-1)$$
$$= (1-t)(-t) = t^2 - t = v,$$

$$\sigma(w) = \sigma\left(\frac{(t^2-t+1)^3}{t^2(t-1)^2}\right) = \frac{\left(\frac{1}{t^2} - \frac{1}{t} + 1\right)^3}{\frac{1}{t^2}\left(\frac{1}{t}-1\right)^2} = \frac{\left(\frac{1}{t^2}\right)^3 (1-t+t^2)^3}{\left(\frac{1}{t^2}\right)^2 (1-t)^2}$$

$$= \frac{(t^2-t+1)^3}{t^2(t-1)^2} = w,$$

and $\tau(w) = \tau\left(\frac{(v+1)^3}{v^2}\right) = \frac{(v+1)^3}{v^2} = w.$

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Let

$$\mathbb{E} = \mathbb{C}(w), \quad \mathbb{F} = \mathbb{C}(t^2 - t) = \mathbb{C}(v)$$

and $\mathbb{G} = \mathbb{C}(t^4) = \mathbb{C}(u)$.

Then

$$\begin{tikzcd}[column sep=small]
 & \mathbb{K} = \mathbb{C}(t) \arrow[dl, dash] \arrow[d, dash] & \\
\mathbb{G} = \mathbb{C}(t^4) & \mathbb{F} = \mathbb{C}(v) \arrow[dr, dash] & \\
 & & \mathbb{E} = \mathbb{C}(w)
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \{1\} \arrow[dl, dash] \arrow[d, dash] & \\
\{1, \rho\} & \{1, \tau\} \arrow[dr, dash] & \\
 & & \{1, \sigma, \tau, \sigma\tau, \tau\sigma, \sigma\tau\sigma\}
\end{tikzcd}$$

and

$$id(t) = t, \qquad \rho(t) = -t,$$

$$\sigma(t) = \frac{1}{t}, \qquad \tau\sigma(t) = \frac{1}{1-t},$$

$$\tau(t) = 1 - t, \qquad \sigma\tau(t) = 1 - \frac{1}{t},$$

$$\sigma\tau\sigma(t) = \frac{1}{1 - \frac{1}{t}} = \frac{t}{t-1} = \frac{-t}{1-t},$$

$$\tau\sigma\tau(t) = 1 - \frac{1}{1-t} = \frac{1-t-1}{1-t} = \frac{-t}{1-t}.$$

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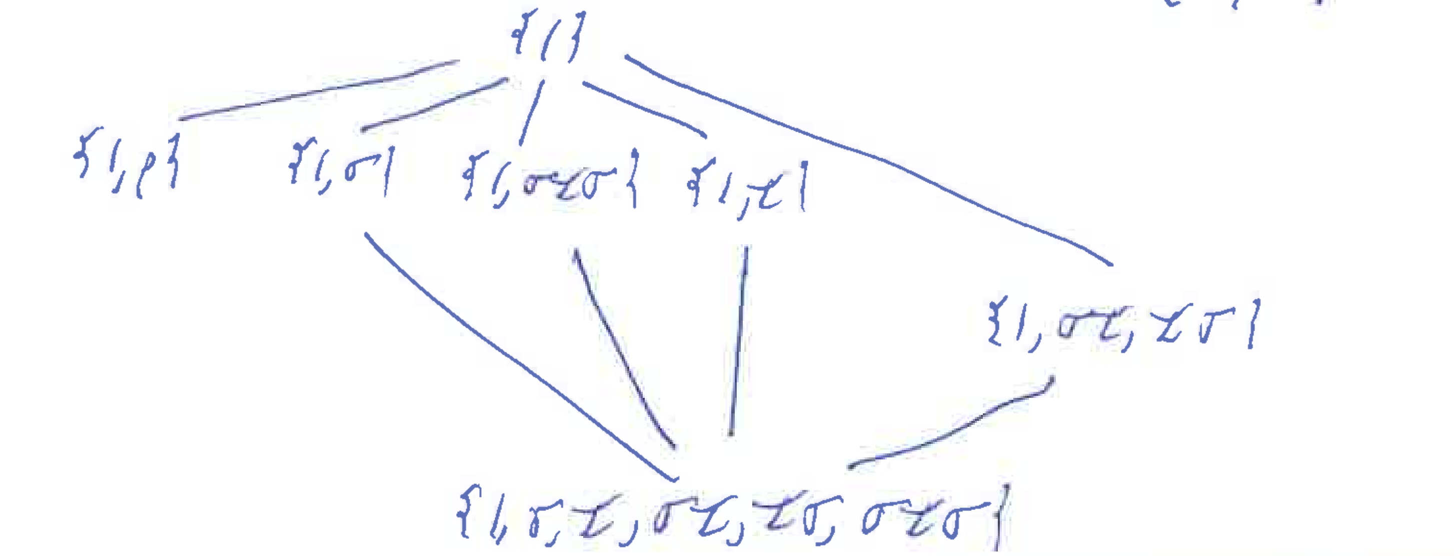
<u>Some minimal polynomials</u>

$$(x-t)(x-\rho(t)) = (x-t)(x+t) = x^2-t^2 = m_{t,\mathbb{Q}(t^2)}(x),$$

$$(x-t)(x-\sigma(t)) = (x-t)\left(x-\frac{1}{t}\right) = x^2-\left(t+\frac{1}{t}\right)x+1$$
$$= m_{t,\mathbb{Q}(t+\frac{1}{t})}(x)$$

$$(x-t)(x-\sigma\tau\sigma(t)) = (x-t)\left(x+\frac{t}{1-t}\right) = x^2+\left(\frac{t}{1-t}-t\right)x-\frac{t^2}{1-t}$$
$$= x^2+\left(\frac{t-1+t^2}{1-t}\right)x-\frac{t^2}{1-t} = x^2+\left(\frac{t^2+t-1}{1-t}\right)x-\frac{t^2}{1-t}$$
$$= x^2+\left(\frac{t^2}{1-t}-1\right)x-\frac{t^2}{1-t} = m_{t,\mathbb{Q}(\frac{t^2}{1-t})}(x)$$

$$(x-t)(x-\sigma\tau(t))(x-\cancel{\tau\sigma}(t)) = (x-t)\left(x-\frac{1}{1-t}\right)\left(x-\left(1-\frac{1}{t}\right)\right)$$
$$= (x-t)\left(x^2-\left(\frac{t-1}{t}+\frac{1}{1-t}\right)x+\frac{1}{t}\right)$$
$$= x^3-\left(\frac{-t^2+2t-1+t}{t(1-t)}\right)x^2+\frac{1}{t}x-tx^2+\left(\frac{-t^2+3t-1}{t-1}\right)x-1$$



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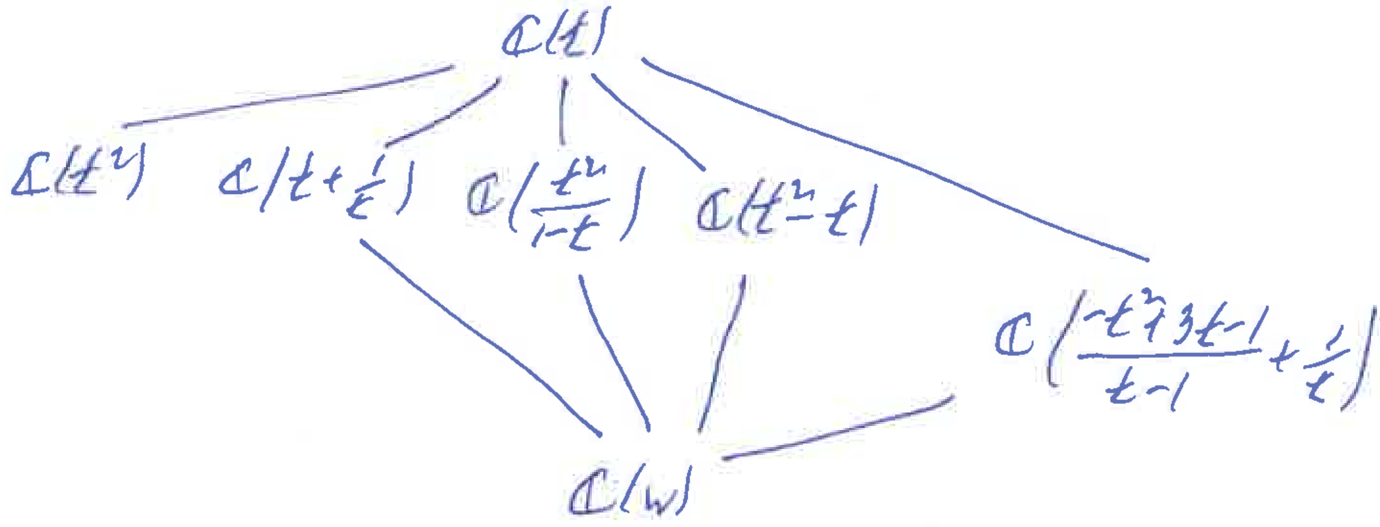
The Hasse diagram of field inclusions is



Let $\zeta = e^{2\pi i/n}$ and $\sigma, \tau \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(t))$ given by

$$\sigma(t) = \zeta t \quad \text{and} \quad \tau(t) = \frac{1}{t}.$$

Then

$$\sigma^n = 1 \quad \text{and} \quad \tau^2 = 1 \quad \text{and} \quad \tau\sigma\tau = \sigma^{-1}$$

since

$$(\tau\sigma\tau)(t) = \tau\sigma\left(\frac{1}{t}\right) = \tau\left(\frac{1}{\zeta t}\right) = \tau\left(\zeta^{-1}\frac{1}{t}\right) = \zeta^{-1}t = \sigma^{-1}(t).$$

So

$$\{1, \sigma, \sigma^2, \dots, \sigma^{n-1}, \tau, \sigma\tau, \dots, \sigma^{n-1}\tau\}$$

is a dihedral group of order $2n$.

$$(x-t)(x-\zeta t)(x-\zeta^2 t)\cdots(x-\zeta^{n-1}t) = x^n - t^n \quad \text{and}$$
$$(x-t^{-1})(x-\zeta t^{-1})(x-\zeta^2 t^{-1})\cdots(x-\zeta^{n-1}t^{-1}) = x^n - t^{-n}.$$

Then

$$m_{t,\mathbb{C}(t^n+t^{-n})}(x) = (x^n - t^n)(x^n - t^{-n}) = x^{2n} - (t^n + t^{-n})x^n + 1.$$

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Note that

$$t^2(t-1)^2 w = (t^2-t+1)^3$$

so that

$(x+1)^3 - x^4 w$ has $v = t^2 - t$ as a root

So $$m_{v,\mathbb{C}(w)}(x) = x^3 + (3-w)x^2 + 3x + 1.$$

The splitting field of $x^3 + (3-w)x^2 + 3x + 1$ over $\mathbb{C}(w)$ is $\mathbb{C}(t)$, with $\dim_{\mathbb{C}(w)}(\mathbb{C}(t)) = 6$.

Then

$$\mathbb{C}(t)^{\{1,\sigma\tau\sigma\}} = \sigma(\mathbb{C}(t)^{\{1,\tau\}}) = \sigma(\mathbb{C}(t^2-t))$$

$$= \mathbb{C}\left(\tfrac{1}{t^2} - \tfrac{1}{t}\right),$$

$$\mathbb{C}(t)^{\{1,\sigma\}} = \tau(\mathbb{C}(t)^{\{1,\tau\sigma\tau\}}) = \tau(\mathbb{C}(t)^{\{1,\sigma\tau\sigma\}})$$

$$= \tau\left(\mathbb{C}\left(\tfrac{1}{t^2} - \tfrac{1}{t}\right)\right) = \mathbb{C}\left(\tfrac{1}{(1-t)^2} - \tfrac{1}{1-t}\right)$$

$$= \mathbb{C}\left(\frac{1+t-1}{(1-t)^2}\right) = \mathbb{C}\left(\frac{t}{(1-t)^2}\right)$$