Finitely generated and free modules Algebra Lect. 22

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Let R be a ring and $k \in \mathbb{Z}_{>0}$.

$$R^{\oplus k} = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} \mid a_i \in R \right\} \text{ with } \begin{array}{ccc} R \times R^{\oplus k} & \to & R^{\oplus k} \\ \left(r, \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix}\right) & \mapsto & \begin{pmatrix} ra_1 \\ \vdots \\ ra_k \end{pmatrix} \end{array}$$

is an R-module. Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \dots, \ e_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Let M be an R-module.

The module M is <u>finitely generated</u> if there exists $k \in \mathbb{Z}_{>0}$ and $b_1, \dots, b_k \in M$ such that the R-module homomorphism given by

$$\begin{array}{ccc} R^{\oplus k} & \to & M \\ e_i & \mapsto & b_i \end{array} \quad \text{is surjective.}$$

Let $K = \ker\varphi$. Then M is given by generators $b_1, \dots, b_k$ and relations

if $k \in K$ and $k = r_1 e_1 + \dots + r_k e_k$

then $r_1 b_1 + \dots + r_k b_k = 0$ in M.

$$M \cong \frac{R^{\oplus k}}{K}.$$

The module M is free of rank k if the R-module homomorphism given by

$$\begin{array}{ccc} R^{\oplus k} & \xrightarrow{\varphi} & M \\ e_i & \longmapsto & b_i \end{array} \quad \text{is bijective.}$$

If M is free of rank then $K = \ker(\varphi)$ is 0 and M is generated by $\{b_1, \ldots, b_k\}$ with no relations. In other words $\{b_1, \ldots, b_k\}$ is a basis of M.

Let M be an R-module,

$$\mathrm{ann}(M) = \{r \in R \mid \text{if } m \in M \text{ then } rm = 0\}.$$

$$\mathrm{Tor}(M) = \left\{ m \in M \;\middle|\; \begin{array}{l} \text{there exists } r \in R \text{ with} \\ r \neq 0 \text{ and } rm = 0 \end{array} \right\}$$

Proposition

(a) $\mathrm{ann}(M)$ is an ideal of R

(b) $\mathrm{Tor}(M)$ is an R-submodule of M.

(c) $\mathrm{ann}(M_1 \oplus M_2) = \mathrm{ann}(M_1) \cap \mathrm{ann}(M_2)$

(d) $\mathrm{Tor}(M_1 \oplus M_2) = \mathrm{Tor}(M_1) \oplus \mathrm{Tor}(M_2)$.

Torsion free versus free

The R-module M is <u>torsion free</u> if

$$\mathrm{Tor}(M) = 0.$$

<u>Proposition</u> If M is a free R-module then $\mathrm{Tor}(M) = 0$.

<u>Example</u> Let R be a PID.
Let M be a finitely generated R-module.
Then there exist $\ell, k \in \mathbb{Z}_{>0}$ and $d_1, \dots, d_\ell \in R$ such that

$$d_1 R \supseteq d_2 R \supseteq \cdots \supseteq d_\ell R \quad \text{and}$$

$$M \cong R^{\oplus k} \oplus \frac{R}{d_1 R} \oplus \cdots \oplus \frac{R}{d_\ell R}.$$

Then

$$\mathrm{ann}(M) = \begin{cases} 0, & \text{if } k \neq 0, \\ d_\ell R, & \text{if } k = 0, \end{cases} \quad \text{and}$$

$$\mathrm{Tor}(M) = \frac{R}{d_1 R} \oplus \cdots \oplus \frac{R}{d_\ell R}$$

Then M is free if $\ell = 0$ and

M is torsion free if $\ell = 0$.

<u>Examples</u>

(1) As a $\mathbb{Z}$-module, $\mathbb{Q}$ is torsion free but not free.

(2) Let $R = \mathbb{R}[x,y]$ and $I = Rx + Ry$ so that I is the ideal of R generated by x and y. Then I is a submodule of the free R-module $R^{\oplus 1} = R$ but I is not free.

(3) Let $R = \frac{\mathbb{Z}}{6\mathbb{Z}}$ and $N = \{(0,0), (3,0)\}$ which is a submodule of $R^{\oplus 2}$. Then N is a submodule of a free module that is not free.

(4) The $\mathbb{Z}$-modules, $2\mathbb{Z}$ and $\mathbb{Z}$ are both free, both of rank 1 and $2\mathbb{Z} \subsetneq \mathbb{Z}$.

<u>Theorem</u> If R is a PID and K is an R-submodule of a free R-module $R^{\oplus k}$ then K is a free R-module.