Constructible numbers

A <u>constructible number</u> is $a \in \mathbb{R}$ such that there exists $k \in \mathbb{Z}_{>0}$ and field inclusions

$$\mathbb{Q} \subseteq \mathbb{E}_1 \subseteq \cdots \subseteq \mathbb{E}_k \subseteq \mathbb{R}$$

such that if $i \in \{1, \dots, k\}$ then there exists $r_i \in \mathbb{E}_i$ such that $\mathbb{E}_{i+1} = \mathbb{E}_i(\sqrt{r_i})$.

In other words,

$$\mathbb{E}_k = \mathbb{Q}(\sqrt{r_1}, \dots \sqrt{r_k}) \text{ with } r_i \in \mathbb{Q}(\sqrt{r_1}, \dots \sqrt{r_{i-1}})$$

Let

$$\mathbb{R}_{const} = \left\{ \alpha \in \mathbb{R} \;\middle|\; \alpha \text{ is a constructible number} \right\}$$

<u>Theorem</u> (a) $\mathbb{R}_{const}$ is a field.

(b) If $a \in \mathbb{R}_{const}$ then $\sqrt{a} \in \mathbb{R}_{const}$

$\mathbb{R}_{const}$ contains all elements in all fields in towers of the form

$$\begin{array}{c} \mathbb{Q}(\sqrt{r_1}, \dots, \sqrt{r_k}) \\ | \\ \vdots \\ | \\ \mathbb{Q}(\sqrt{r_1}, \sqrt{r_2}) \\ | \\ \mathbb{Q}(\sqrt{r_1}) \\ | \\ \mathbb{Q}. \end{array}$$

Constructible points

Theorem A point $(a,b) \in \mathbb{R}^2$ is constructible by ruler and compass if and only if

$$a \in \mathbb{R}_{const} \text{ and } b \in \mathbb{R}_{const}.$$

Things we CANNOT construct

(a) A $20°$ angle: When $3\theta = 60°$ then

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$$

gives $\frac{1}{2} = 4\alpha^3 - 3\alpha$ for $\alpha = \cos(20°)$

The polynomial $8x^3 - 3x - 1$ is irreducible in $\mathbb{Q}[x]$. So $\alpha \in \mathbb{R}$ is an element of $\mathbb{R}$ that does not lie in an extension of $\mathbb{Q}$ of degree 2^m.

(b) A regular p-gon, where $p \in \mathbb{Z}_{>0}$ is prime and $p \neq 2^m+1$ with $m \in \mathbb{Z}_{>0}$.
The minimal polynomial of $e^{2\pi i/p}$ is

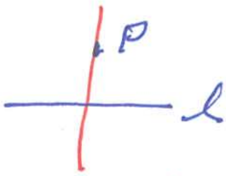
$\Phi_p(x) = x^{p-1} + \cdots + x + 1$, which has $\deg(\Phi_p(x)) = p-1$ and is irreducible. Since $p-1$ is not a power of 2, then $e^{2\pi i/p}$ is not constructible in $\mathbb{R}^2$. More specifically a regular 7-gon is not constructible.

Ruler and compass construction

Start with:

(0,0) (1,0).

(C1) From 

construct 

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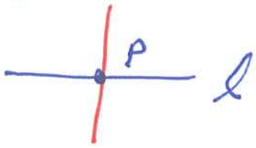
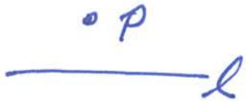
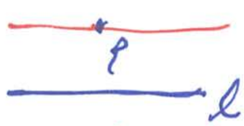
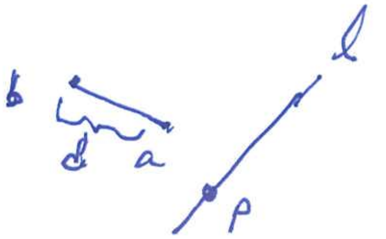
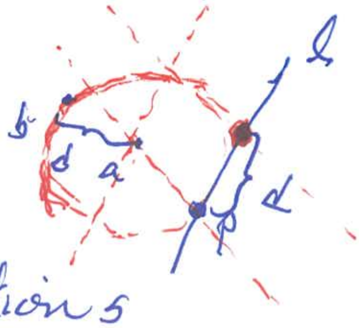
construct 

(C3) From 

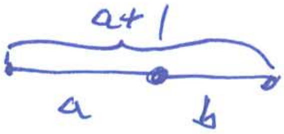
construct 

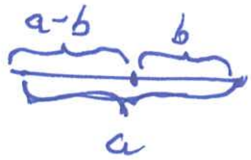
(C4) From 

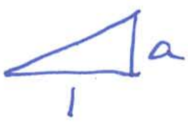
construct 

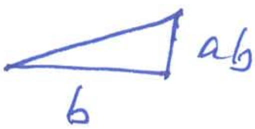
Ruler and compass field operations

addition: and subtraction:

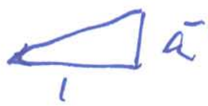




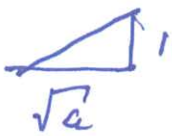
multiplication: 

is similar to 

division: 

is similar to 

square roots: 

is similar to 

and embed these in a circle of diameter a+1