07.05.2024 (1)
Algebra Lect 29
A. Ram.

Rational canonical form versus Jordan normal form.

The matrix $T = \begin{pmatrix} 5 & -4 & 1 \\ -1 & -1 & -2 \\ 3 & 0 & 3 \end{pmatrix} \in M_{3\times 3}(\mathbb{Q})$

defines a $\mathbb{Q}[x]$-module by

$$\mathbb{Q}[x] \times \mathbb{Q}^3 \longrightarrow \mathbb{Q}^3$$

$$(a_0 + a_1 x + \cdots + a_\ell x^\ell, v) \longmapsto (a_0 + a_1 A + \cdots + a_\ell A^\ell)v.$$

This $\mathbb{Q}[x]$-module $\mathbb{Q}^3$ has generators

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and relation matrix

$$A = \begin{pmatrix} x-5 & +4 & -1 \\ +1 & x+1 & +2 \\ -3 & 0 & x-3 \end{pmatrix}$$

Smith normal form produces $P, Q \in GL_3(\mathbb{Q}[x])$

and $D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$ with $d_1, d_2, d_3 \in \mathbb{Q}[x]$

such that

$$d_1 \mathbb{Q}[x] \supseteq d_2 \mathbb{Q}[x] \supseteq d_3 \mathbb{Q}[x].$$

and $PDQ = A.$

The determinant of A is

$$-3(8+(x+1))+(x-3)((x-5)(x+1)-4)$$

$$= -24-3x-3+(x-3)(x^2-4x-5-4)$$

$$= -3x-27+(x-3)(x^2-4x-9)$$

$$\begin{aligned} = & -27-3x \\ & +27+12x-3x^2 \\ & \qquad -9x-4x^2+x^3 \end{aligned}$$

$$= x^3-7x^2=(x-7)x^2.$$

Then

$$\gcd\left\{\begin{matrix} x-5, & 4, & -1 \\ 1 & x+1, & 2 \\ -3, & 0, & x-3 \end{matrix}\right\}=1 \quad \text{and} \quad \gcd\left\{\begin{matrix} (x+1)(x-3), & \cdot \ \cdot & , \\ \cdot \ - , \ \cdot & & ,12 \\ - \ + & - ,- & \cdot \end{matrix}\right\}=1$$

So, if

$$D=\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \quad \text{then} \quad \begin{aligned} d_1 &= 1 \\ d_1 d_2 &= 1 \\ d_1 d_2 d_3 &= x^2(x-7) \end{aligned}$$

which gives $d_1=1$, $d_2=1$, $d_3=x^2(x-7)$

and

$$D=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^2(x-7) \end{pmatrix}$$

The Smith normal form provides a change of generators to give that, as $\mathbb{Q}[x]$-modules,

$$\mathbb{Q}^3 \cong \frac{\mathbb{Q}[x]}{1\cdot\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{1\cdot\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x-7)x^2\mathbb{Q}[x]} \cong \frac{\mathbb{Q}[x]}{x^2(x-7)\mathbb{Q}[x]}$$

Since $x^2(x-7) = x^3 - 7x^2 = 0$ in $\frac{\mathbb{Q}[x]}{x^2(x-7)\mathbb{Q}[x]}$ then the matrix of x acting on $\frac{\mathbb{Q}[x]}{x^2(x-7)\mathbb{Q}[x]}$ with respect to the basis $\{1, x, x^2\}$ is

$$R = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -7 \end{pmatrix}$$ which is the rational canonical form of T.

Using Chinese block decomposition, as $\mathbb{Q}[x]$-modules

$$\mathbb{Q}^3 \cong \frac{\mathbb{Q}[x]}{(x-7)\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{x^2\mathbb{Q}[x]}$$

and the matrix of x acting on $\frac{\mathbb{Q}[x]}{(x-7)\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{x^2\mathbb{Q}[x]}$ with respect to the basis $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\x\end{pmatrix}\right\}$ is

$$S = \left(\begin{array}{c|cc} 7 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right),$$ which is the Jordan normal form of T.

the sources of logical constructs

(1) "x is finite" captures the statement

$$x \in \mathbb{Z}_{\geq 0}$$

(2) "There exists $s \in S$ such that $s < t$" captures the statement

$$\{x \in S \mid x < t\} \neq \emptyset$$

or $\mathrm{Card}\{x \in S \mid x < t\} \neq 0$

(3) "There exists a unique $s \in S$ such that $s < t$" captures the statement

$$\{x \in S \mid x < t\} = \{s\}$$

or $\mathrm{Card}\{x \in S \mid x < t\} = 1$.

(4) The proof by induction mechanics:

Base case: $n = 1$, ...

Base case: $n = 2$, ...

Base case: $n = 3$, ...

Induction step: ...

captures the definition of $\mathbb{Z}_{>0}$.

Let $n \in \mathbb{Z}_{>0}$.

An $n\times n$ permutation matrix is an $n \times n$ matrix such that

(a) there is exactly one nonzero entry in each row and each column,

(b) the nonzero entries are 1.

The symmetric group S_n is

$$S_n = \{ n\times n \text{ permutation matrix} \}.$$

Let $c \in S_n$ be the permutation matrix given by

$$c_{n,1} = 1 \quad \text{and} \quad c_{i,i+1}, \text{ for } i \in \{1, \ldots, n-1\}.$$

If $n=5$ then

$$c = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $w_0 \in S_n$ be the permutation matrix given by

$$(w_0)_{i,n-i+1} = 1, \quad \text{for } i \in \{1, \ldots, n\}.$$

If $n=5$ then

$$w_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

07.05.2014 (6)

Algebra Lect 29

A. Ram

A cyclic matrix is an element of

$$C_n = \{1, c, c^2, \ldots, c^{n-1}\}$$

A dihedral matrix is an element of

$$D_n = \{1, w_0, c, cw_0, \ldots, c^{n-1}, c^{n-1}w_0\}.$$