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Isomorphisms and automorphisms

An <u>R-algebra isomorphism</u> is an R-algebra morphism $\varphi: A \to B$ such that there exists an R-algebra morphism $\psi: B \to A$ such that $\psi \circ \varphi = id_A$ and $\varphi \circ \psi = id_B$.

A <u>automorphism</u> of an R-algebra A is an R-algebra isomorphism $\varphi: A \to A$

A <u>Gset isomorphism</u> is a G-set morphism $\varphi: S \to T$ such that there exists a G-set morphism $\psi: T \to S$ such that

$\psi \circ \varphi = id_S$ and $\varphi \circ \psi = id_T$.

An <u>automorphism</u> of a G-set A is a G-set isomorphism $\varphi: A \to A$.

The category of sets

A <u>bijection</u> is a function $\varphi: A \to B$ such that

there exists a function $\psi: B \to A$

such that $\psi \circ \varphi = id_A$ and $\varphi \circ \psi = id_B$.

A <u>permutation</u> of a set S is a

bijection $\varphi: S \to S$

<u>The rank-nullity theorem</u> Let $\mathbb{F}$ be a field. Let $A \in M_{t \times s}(\mathbb{F})$. Then

$$\text{rank}(A) = s - \text{nullity}(A).$$

A better statement is that an

$\mathbb{F}$-linear transformation $T: \mathbb{F}^s \to \mathbb{F}^t$

satisfies

$$\text{rank}(T) = s - \text{nullity}(T).$$

A better statement is

$$\dim(\text{im}(T)) = \dim(\mathbb{F}^s) - \dim(\ker(T)).$$

A better statement is

$$\dim(\text{im}(T)) = \dim\left(\mathbb{F}^s / \ker(T)\right).$$

A better statement is
as $\mathbb{F}$-modules,

$$\mathrm{im}(H) \cong \frac{\mathbb{F}^5}{\ker(H)}.$$

<u>The orbit-stabilizer theorem</u>

Let G be a group and let X be a G-set.
Let $x \in X$. The <u>orbit of x</u> is

$$Gx = \{gx \mid g \in G\} \quad \text{and}$$

$$\mathrm{Stab}_G(x) = \{g \in G \mid gx = x\}.$$

is the <u>stabilizer of x</u>. The orbit-stabilizer theorem says

$$\mathrm{Card}(\mathrm{Stab}_G(x)) \cdot \mathrm{Card}(Gx) = \mathrm{Card}(G).$$

Let $H = \mathrm{Stab}_G(x)$. Then

$$\mathrm{Card}(Gx) = \frac{\mathrm{Card}(G)}{\mathrm{Card}(H)} = \mathrm{Card}(G/H)$$

Let $\varphi : G \to X$ be the G-set morphism
given by

$$\begin{aligned} \varphi : G &\to X \\ g &\mapsto gx. \end{aligned}$$

Then
$$\mathrm{im}(\varphi) = Gx.$$

So a better statement of the orbit stabilizer theorem is

$$\mathrm{Card}(\mathrm{im}\,\varphi) = \mathrm{Card}(G/H).$$

An even better statement is

as G-sets $\quad \mathrm{im}(\varphi) \simeq \dfrac{G}{\mathrm{Stab}_G(x)}.$

<u>What is a category?</u>

A category is determined by its objects, morphisms and compositions.

A <u>category $\mathcal{C}$</u> is a collection of sets and functions:

(1) A set $\mathrm{Obj}(\mathcal{C})$,

(2) A set $\mathrm{Hom}(X,Y)$ for $X, Y \in \mathrm{Obj}(\mathcal{C})$,

(3) A function

$$\begin{array}{ccc} \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) & \longrightarrow & \mathrm{Hom}(X,Z) \\ (f\ ,\ g) & \longmapsto & gf \end{array}$$

for $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$

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such that

(a) $\mathrm{Hom}(X,X)$ contains an identity id_X,

(b) The composition maps satisfy associativity

Examples of categories

Category	Objects	Morphisms
Sets	Sets	functions
Groups	Groups	group morphisms
Rings	Rings	ring morphisms.
Vector spaces	Vector spaces	linear transformations
R-modules	R-modules	R-module morphisms
abelian groups	abelian groups	abelian group morphisms
commutative rings	commutative rings	ring morphisms
fields	fields	ring morphisms
topological spaces	topological spaces	continuous functions.

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Category	Objects	Morphisms
uniform spaces	uniform spaces	uniformly continuous functions
varieties	varieties	morphisms
schemes	schemes	morphisms
sheaves	sheaves	morphisms
categories	categories	natural transformations.