2.2 Proof that the poset of R-modules is a modular lattice

Proposition 2.2. *Let R be a ring and let M be an R-module. Let N be an R-submodule of M. Define*

$$\mathcal{S}_{[N,M]} = \{P \mid N \subseteq P \subseteq M \text{ are } R\text{-module inclusions}\} \qquad \text{partially ordered by inclusion.}$$

For $P, Q \in \mathcal{S}_{[N,M]}$, define

$$P + Q = \{p + q \mid p \in P \text{ and } q \in Q\} \qquad \text{and} \qquad P \cap Q = \{m \in M \mid m \in P \text{ and } m \in Q\}$$

(a) Let $P, Q \in \mathcal{S}_{[N,M]}$. Then

$$P + Q = \sup(P, Q) \qquad \text{and} \qquad P \cap Q = \inf(P, Q).$$

(b) (modular law) If $L, P, Q \in \mathcal{S}_{[N,M]}$ and $P \subseteq Q$ then $Q + (L \cap P) = (Q + L) \cap P$.

Proof.
(a) To show: (aa) $P \subseteq P + Q$ and $Q \subseteq P + Q$.
(ab) If $L \in \mathcal{S}_{[N,M]}$ and $P \subseteq L$ and $Q \subseteq L$ then $P + Q \subseteq L$.
(ac) $P \cap Q \subseteq P$ and $P \cap Q \subseteq Q$.
(ad) If $K \in \mathcal{S}_{[N,M]}$ and $K \subseteq P$ and $K \subseteq Q$ then $K \subseteq P \cap Q$.
(b) To show: If $P \subseteq Q$ then $Q \cap (P + L) = P + (Q \cap L)$. Assume $P \subseteq Q$.
To show: $Q \cap (P + L) = P + (Q \cap L)$.
To show: (ba) $Q \cap (P + L) \subseteq P + (Q \cap L)$.
To show: (bb) $P + (Q \cap L) \subseteq Q \cap (P + L)$.

(ba) Assume $a \in Q \cap (P + L)$.
To show: $a \in P + (Q \cap L)$.
So there exist $p \in P$ and $\ell \in L$ such that $a = p + \ell$.
Since $a \in Q$ and $p \in Q$ then $\ell = a - p \in Q$.
So $\ell \in Q \cap L$.
So $a = p + \ell \in P + (Q \cap L)$.
So $Q \cap (P + L) \subseteq P + (Q \cap L)$.
(bb) Assume $b \in P + (Q \cap L)$.
To show: $b \in Q \cap (P + L)$
Since $b \in P + (Q \cap L)$ then there exist $p \in P$ and $\ell \in Q \cap L$ such that $b = p + \ell$.
Since $P \subseteq Q$ then $p \in Q$.
So $b = p + \ell \in Q \cap (P + L)$.
So $P + (Q \cap L) \subseteq Q \cap (P + L)$.

$P + (Q \cap L) = Q \cap (P + L)$. □