### 19.6.3 Smith Normal form

110. Determine the Jordan normal form of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ by calculating the invariant factor matrix of $X - A$.

111. Find all possible Jordan normal forms for a matrices with characteristic polynomial $(t+2)^2(t-5)^3$.

112. Find the Smith normal form of $A = \begin{pmatrix} 5 & -4 & 1 \\ -1 & -1 & -2 \\ 3 & 0 & 3 \end{pmatrix}$ over $\mathbb{Z}$.

113. Find the rational canonical form of $A = \begin{pmatrix} 5 & -4 & 1 \\ -1 & -1 & -2 \\ 3 & 0 & 3 \end{pmatrix}$ over $\mathbb{Q}$.

114. Find the Jordan canonical form of $A = \begin{pmatrix} 5 & -4 & 1 \\ -1 & -1 & -2 \\ 3 & 0 & 3 \end{pmatrix}$ over $\mathbb{C}$.

115. Find the Smith normal form of $\begin{pmatrix} 11 & -4 & 7 \\ -1 & 2 & 1 \\ 3 & 0 & 3 \end{pmatrix}$ over $\mathbb{Z}$.

116. Let $A = \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$. Find $L, R \in GL_3(\mathbb{Z})$ and $d_1, d_2, d_3 \in \mathbb{Z}_{\geq 0}$ such that $d_3\mathbb{Z} \subseteq d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $LAR = \mathrm{diag}(d_1, d_2, d_3)$.

117. Let $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$. Find $L, R \in GL_2(\mathbb{Z})$ and $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ such that $d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $LAR = \mathrm{diag}(d_1, d_2)$.

118. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Find $L \in GL_2(\mathbb{Z})$ and $R \in GL_3(\mathbb{Z})$ and $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ such that $d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $LAR = \mathrm{diag}(d_1, d_2)$.

119. Let $A = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}$. Find $L, R \in GL_3(\mathbb{Z})$ and $d_1, d_2, d_3 \in \mathbb{Z}_{\geq 0}$ such that $d_3\mathbb{Z} \subseteq d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $LAR = \mathrm{diag}(d_1, d_2, d_3)$.

120. Let $R = \mathbb{Q}[X]$. Let $A = \begin{pmatrix} 1-X & 1+X & X \\ X & 1-X & 1 \\ 1+X & 2X & 1 \end{pmatrix}$. Find $P, Q \in GL_3(R)$ and $d_1, d_2, d_3 \in \mathbb{Q}[X]_{\mathrm{monic}}$ such that $d_3R \subseteq d_2R \subseteq d_1R$ and $PAQ = \mathrm{diag}(d_1, d_2, d_3)$.

121. Let $R = \mathbb{Q}[X]$. Let $A = \begin{pmatrix} X & 1 & -2 \\ -3 & X+4 & -6 \\ -2 & 2 & X-3 \end{pmatrix}$. Find $P, Q \in GL_3(R)$ and $d_1, d_2, d_3 \in \mathbb{Q}[X]_{\mathrm{monic}}$ such that $d_3R \subseteq d_2R \subseteq d_1R$ and $PAQ = \mathrm{diag}(d_1, d_2, d_3)$.

122. Let $R = \mathbb{Q}[X]$. Let $A = \begin{pmatrix} X & 0 & 0 \\ 0 & 1-X & 0 \\ 0 & 0 & 1-X^2 \end{pmatrix}$. Find $P, Q \in GL_3(R)$ and $d_1, d_2, d_3 \in \mathbb{Q}[X]_{\mathrm{monic}}$ such that $d_3R \subseteq d_2R \subseteq d_1R$ and $PAQ = \mathrm{diag}(d_1, d_2, d_3)$.

123. Let $X$ be a $n \times m$ matrix with entries in a ring $R$. Define an ideal $d_1(X)$ to be the ideal in $R$ generated by all entries of $X$. Let $A$ and $B$ be invertible matrices (of the appropriate sizes) with entries in $R$. Prove that $d_1(AXB) = d_1(X)$.

124. With notation as in Question 123, let $d_k(X)$ be the ideal in $R$ generated by all $k \times k$ minors in $X$. Prove that $d_k(AXB) = d_k(X)$.

125. Use the previous result to show that the elements $d_i$ in Smith Normal Form are unique up to associates.