2.23 Proof of the properties of $\mathrm{Tor}(M)$

Proposition 2.29. *Let R be an integral domain and let M be an R-module.*

(a) *If M is an R-module then $\mathrm{Tor}(M)$ is an R-submodule of M.*

(b) *If M and N are R-modules then $\mathrm{Tor}(M \oplus N) = \mathrm{Tor}(M) \oplus \mathrm{Tor}(N)$.*

(c) $\mathrm{Tor}(R) = 0$.

(d) *If $d \in R$ and $d \neq 0$ then $\mathrm{Tor}(R/dR) = R/dR$.*

Proof.
(a) To show: (aa) If $m_1, m_2 \in \mathrm{Tor}(M)$ then $m_1 + m_2 \in \mathrm{Tor}(M)$,
(ab) If $c \in R$ and $m \in \mathrm{Tor}(M)$ then $cm \in \mathrm{Tor}(M)$,

(aa) Assume $m_1, m_2 \in \mathrm{Tor}(M)$.
To show: $m_1 + m_2 \in \mathrm{Tor}(M)$.
Since $m_1, m_2 \in M$ there exists $a_1, a_2 \in R$ such that $a_1 \neq 0$ and $a_2 \neq 0$ and $a_1 m_1 = 0$ and $a_2 m_2 = 0$.
Since R is an integral domain $a_1 a_2 \neq 0$.
Then $a_1 a_2 (m_1 + m_2) = a_2 a_1 m_1 + a_1 a_2 m_2 = 0 + 0 = 0$.
So $m_1 + m_2 \in \mathrm{Tor}(M)$.

(ab) Assume $c \in R$ and $m \in \mathrm{Tor}(M)$.
Since $m \in \mathrm{Tor}(M)$ there exists $a \in R$ such that $a \neq 0$ and $am = 0$.
Then $a \cdot cm = cam = c \cdot 0 = 0$.
So $cm \in \mathrm{Tor}(M)$.

(b) To show: (ba) If $(m, n) \in \mathrm{Tor}(M \oplus N)$ then $m \in \mathrm{Tor}(M)$ and $n \in \mathrm{Tor}(N)$.
(bb) If $m \in \mathrm{Tor}(M)$ and $n \in \mathrm{Tor}(N)$ then $(m, n) \in \mathrm{Tor}(M \oplus N)$.

(ba) Assume $(m, n) \in \mathrm{Tor}(M \oplus N)$.
Then there exists $a \in \mathbb{A}$ such that $a \neq 0$ and $a(m, n) = (am, an) = (0, 0)$.
So $a \neq 0$ and $am = 0$ and $an = 0$.
So $m \in \mathrm{Tor}(M)$ and $n \in \mathrm{Tor}(N)$ and $(m, n) \in \mathrm{Tor}(M) \oplus \mathrm{Tor}(N)$.

(bb) Assume $(m, n) \in \mathrm{Tor}(M) \oplus \mathrm{Tor}(N)$.
Then $m \in \mathrm{Tor}(M)$ and $n \in \mathrm{Tor}(N)$.
Then there exists $a_1, a_2 \in \mathbb{A}$ such that $a_1 m = 0$ and $a_2 n = 0$.
Let $a = a_1 a_2$.
Then $a(m, n) = (a_1 a_2 m, a_1 a_2 n) = (a_2 a_1 m, a_1 a_2 n) = (a_2 \cdot 0m, a_1 \cdot 0n) = (0, 0)$.
So $(m, n) \in \mathrm{Tor}(M \oplus N)$.

(c) To show: If $r \in \mathrm{Tor}(R)$ then $r = 0$.
Assume $r \in \mathrm{Tor}(R)$.
Then there exists $a \in R$ such that $a \neq 0$ and $ar = 0$.
Since R is an integral domain and $ar = 0$ and $a \neq 0$ then $r = 0$.
So $\mathrm{Tor}(R) = 0$.

(d) Assume $d \in R$ and $d \neq 0$.
To show: If $r + dR \in R/dR$ then $r + dR \in \mathrm{Tor}(R/dR)$.
Assume $r + dR \in R/dR$.

To show: $r + dR \in \mathrm{Tor}(R/dR)$.
To show: There exists $a \in R$ such that $a \neq 0$ and $a(r + dR) = 0 + dR$. Let $a = d$.
Since $d \neq 0$ then $a \neq 0$ and since $dr \in dR$ then $a(r + dR) = dr + dR = 0 + dR$.
So $\mathrm{Tor}(R/dR) = R/dR$. □