1.12 Tutorial 10 MAST30005 Semester I, 2024: Rings, homorphisms, units

1. Let R be a ring and let $a, b, c \in R$. Show that
   (a) if $a + b = a + c$ then $b = c$,
   (b) $a0 = 0 = 0a$,
   (c) $a(-b) = (-a)b = -(ab)$,
   (d) $(-a)(-b) = ab$.

2. True or false? (Be sure to include a proof.)
   (a) Every field is also a ring.
   (b) If $R$is a commutative ring and $a, b, c \in R$ and $ac = bc$ then $a - b$.
   (c) The nonzero elements in a ring form a group under multiplication.

3. Let $\xi = (-1 + \sqrt{-3})/2 \in \mathbb{C}$. Using $\xi^2 + \xi + 1 = 0$, show that the *Eisenstein integers*
$$\mathbb{Z}[\xi] = \{a + b\xi \mid a, b \in \mathbb{Z}\} \qquad \text{is a subring of } \mathbb{C}.$$
   Does there exist a homomorphism from $\mathbb{Z}[\xi]$ to $\mathbb{F}_2$?

4. Determine,
$$\mathbb{Z}^\times, \qquad (\mathbb{Z}/5\mathbb{Z})^\times, \qquad (\mathbb{Z}/15\mathbb{Z})^\times, \qquad \mathbb{Q}^\times, \qquad \mathbb{Q}[x]^\times, \qquad \mathbb{Z}[i]^\times, \qquad \text{and} \qquad \mathbb{Z}[\sqrt{2}^\times.$$

5. Let I be an ideal of a ring R. Show that if I contains a unit of R then $I = R$.

6. (a) Let $f = x^2$ and $d = 2x + 1$. Find $q, r \in \mathbb{Q}[x]$ such that $f = qd + r$ and $\deg r < \deg d$.
   (b) Show that $\mathbb{Q}[x]$ is a Euclidean domain with respect to the degree function.
   (c) Show that $\mathbb{Z}[x]$ is not a Euclidean domain with respect to the degree function.
   (d) Show that $\mathbb{Q}[x]$ is a PID.
   (e) Show that $\mathbb{Z}[x]$ is not a PID.
   (f) Explain exactly where the proof that $\mathbb{Q}[x]$ is a PID fails to show that $\mathbb{Z}[x]$ is PID.

7. Let $\zeta = e^{2\pi i/3}$. Show that the Eisenstein integers $\mathbb{Z}[\zeta]$ with the function
$$N\colon \begin{array}{ccc} \mathbb{Z}[\zeta] & \to & \mathbb{Z}_{\ge 0} \\ z & \mapsto & |z|^2 \end{array}$$
   is a Euclidean domain.

8. Let $R = \mathbb{R}[x]$.
   (a) Show that $\mathbb{R}[y]$ with the degree function is a Euclidean domain.
   (b) Show that $\mathbb{R}[y]$ is a PID.
   (c) Show that $R[y]$ with the degree function is not a Euclidean domain.

(d) Show that $R[y]$ is not a PID.

(e) Explain exactly where the proof that $\mathbb{R}[y]$ is a PID fails to show that $R[y]$ is a PID.

9. Let $\phi\colon \mathbb{R}[x] \to \mathbb{C}$ be the $\mathbb{R}$-linear transformation given by $\phi(x) = i$. Determine $\ker(\phi)$ and show that

$$\mathbb{R}[x]/(X^2+1) \cong \mathbb{C} \qquad \text{as rings.}$$

10. Show that every ideal in $\mathbb{Z}/12\mathbb{Z}$ is principal. Is $\mathbb{Z}/12\mathbb{Z}$ a PID?