1.6 Tutorial 4 MAST30005 Semester I, 2024: Generators and relations

1. Show that the symmetric group S_n is presented by generators $s_1, \ldots, s_{n-1}$ and relations
$$s_j^2 = 1, \qquad s_k s_\ell = s_\ell s_k, \qquad s_i, s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$
for $j \in \{1, \ldots, n-1\}$, $j, k \in \{1, \ldots n-1$ with $k \notin \{k+1, k-1\}$ and $i \in \{1, \ldots, n-2\}$.

2. Show that the dihedral group D_n is presented by generators s, r with relations
$$s^2 = 1, \qquad r^n = 1, \qquad sr = r^{-1}s.$$

3. Show that the cyclic group μ_n is presented by a single generator ζ with relation $\zeta^n = 1$.

4. Show that the cyclic group $\mathbb{Z}/n\mathbb{Z}$ is presented by a single generator 1 with relation $n = 0$.

5. Show that the dihedral group D_n is presented by generators s_1, s_2 with
$$s_1^2 = 1, \qquad s_2^2 = 1, \qquad (s_1 s_2)^n = 1.$$

6. Carefully define permutation matrix. Show that the symmetric group S_n is (isomorphic to) the group of $n \times n$ permutation matrices.

7. Carefully define cyclic matrix. Show that the cyclic group μ_n is (isomorphic to) the group of $n \times n$ cyclic matrices.

8. Carefully define dihedral matrices. Show that the dihedral group D_n is (isomorphic to) the group of $n \times n$ dihedral matrices.

9. Show that the symmetric group S_n is (isomorphic to) $\mathrm{Aut}(\{1, \ldots, n\})$.

10. Determine the subgroup lattice of $\mathbb{Z}/2\mathbb{Z}$.

11. Determine the subgroup lattice of $\mathbb{Z}/3\mathbb{Z}$.

12. Determine the subgroup lattice of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

13. Determine the subgroup lattice of $\mathbb{Z}/5\mathbb{Z}$.

14. Show that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

15. Determine the subgroup lattice of $\mathbb{Z}/6\mathbb{Z}$.

16. Show that $S_3 \cong D_3$.

17. Carefully define the quaternion group and determine its subgroup lattice.

18. Show that $\mathbb{C}$ is the $\mathbb{R}$-algebra presented by a single generator i and the relation $i^2 = 1$.

19. Show that $\mathbb{R}[x]$ is the $\mathbb{R}$-algebra presented by a single generator x (and no relations).

20. Show that $\mathbb{Z}$ is the group generated by single generator 1 (and no relations).

21. Show that $\mathbb{R}[x, x^{-1}]$ is the $\mathbb{R}$-algebra presented by a generators x, y with relation $xy = 1$.

22. Show that $\mathbb{F}_4$ is the $\mathbb{F}_2$-algebra presented by a single generator τ with relation $\tau^2 + \tau + 1 = 0$.