1.8 Tutorial 6 Semester I, 2024: Modules and modules over a PID

1. Let M be an R-module. Show that if $r \in R$ and $m \in M$ then

$$0m = 0, \qquad r0 = 0, \qquad \text{and} \qquad (-r)m = -(rm) = r(-m).$$

2. (a) Let $N = \{(x, y, z) \in \mathbb{Z}^3 \mid x + y + z = 0\}$. Is N a free $\mathbb{Z}$-module? If so, find a basis.
   (b) Show that $\mathbb{Q}$ is not free as a $\mathbb{Z}$-module (remember the basis may be infinite).

3. (a) Let V be a vector space over a field k. Let $T : V \to V$ be a linear transformation. Show that defining

$$(\sum_i a_i x^i) \cdot v = \sum_i a_i T^i(v) \qquad \text{makes } V \text{ into a } k[x]\text{-module.}$$

   (b) Find an example of a vector space V, together with two linear transformations T and S, such that there does not exist a $k[x, y]$-module structure on V with such that if $v \in V$ then $x \cdot v = T(v)$ and $y \cdot v = S(v)$ for all $v \in V$.

4. Let M be an R-module and N be a submodule of M. Give (with proof) a natural bijection

$$\{\text{submodules of } M/N \leftrightarrow \text{submodules of } M \text{ containing } N.$$

5. Let R be a principal ideal domain, $p \in R$ an irreducible element, $k \in \mathbb{Z}_{\geq 1}$, let M be the R-module

$$M = R/(p^k) \qquad \text{and let} \qquad N = p^{k-1}M := \{p^{k-1}m \mid m \in M\}.$$

   (a) Show that N is a submodule of M.
   (b) Show that N is contained in every non-zero submodule of M.

6. An R-module iss *cyclic* if it has a generating set with one element.
   (a) Is a quotient of a cyclic module necessarily cyclic?
   (b) Is a submodule of a cyclic module necessarily cyclic?

7. Let V be the $\mathbb{Z}[i]$-module $(\mathbb{Z}[i])^2/N$ where

$$N = \mathrm{span}_{\mathbb{Z}[i]}\{(1 + i, 2 - i), (3, 5i)\}.$$

   Write V as a direct sum of cyclic modules.

8. Let

$$A = \begin{pmatrix} 3 & 8 & 7 & 9 \\ 2 & 4 & 6 & 6 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

   (a) Find the Smith Normal form (over $\mathbb{Z}$) of the matrix A.
   (b) If M is a $\mathbb{Z}$-module with presentation matrix A, then show that $M \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.
   (c) If N is a $\mathbb{Q}$-module with presentation matrix A, identify N.

9. Let V be a two dimensional vector space over $\mathbb{Q}$ having basis $\{v_1, v_2\}$. Let T be the linear operator on V defined by

$$T(v_1) = 3v_1 - v_2 \qquad \text{and} \qquad T(v_2) = 2v_2.$$

Recall that V together with T can be identified with a $\mathbb{Q}[t]$-module by defining $tu = T(u)$ for $u \in V$.

   (a) Show that the subspace $U = \{av_2 \mid a \in \mathbb{Q}\}$ of V is a $\mathbb{Q}[t]$-submodule of V.
   (b) Let $f = t^2 + 2t - 3$. Express fv_1 and fv_2 as linear combinations of v_1 and v_2.

10. (a) Let R be a ring and let M be an R-module. Suppose that U and V are two submodules of M. Show that

$$M \cong U \oplus V \qquad \text{if and only if} \qquad U \cap V = \{0\} \text{ and } U + V = M.$$

   (b) Show that the $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z}$, where p is a prime and n a positive integer, is not a direct sum of two non-zero $\mathbb{Z}$-modules.

11. Up to isomorphism, how many abelian groups of order 96 are there?

12. Find an isomorphic direct sum of cyclic groups, where V is the abelian group generated by x, y, z with relations

$$3x + 2y + 8z = 0 \qquad \text{and} \qquad 2x + 4z = 0.$$

13. Find an isomorphic direct sum of cyclic groups, where V is the abelian group generated by x, y, z with relations

$$x + y = 0, \qquad 2x = 0, \qquad 4x + 2z = 0, \qquad 4x + 2y + 2z = 0.$$

14. Find an isomorphic direct sum of cyclic groups, where V is the abelian group generated by x, y, z with relations

$$2x + y = 0, \qquad \text{and} \qquad x - y + 3z = 0.$$

15. Find an isomorphic direct sum of cyclic groups, where V is the abelian group generated by x, y, z with relations

$$4x + y + 2z = 0, \qquad 5x + 2y + z = 0, \qquad \text{and} \qquad 6y - 6z = 0.$$