## 5.13 Tutorial 7 Semester I, Year 2024: Jordan form and $\mathbb{F}[x]$-modules

1. Let $\mathbb{F}$ be a field and define $D\colon \mathbb{F}[x] \to \mathbb{F}[x]$ by

$$D(a_0 + a_1x + \cdots + a_nx^n) = a_1 + 2a_2x + \cdots + na_nx^{n-1}.$$

   (a) Verify that $D(fg) = D(f)g + fD(g)$, for all $f, g \in \mathbb{F}[x]$.

   (b) An element $\alpha$ is called a double root of $f$ if $(x-\alpha)^2$ divides $f$. Prove that $\alpha$ is a double root of $f$ if and only if $f(\alpha) = 0$ and $(Df)(\alpha) = 0$.

2. Let $E = \mathbb{Q}(\alpha)$, where $\alpha^3 - \alpha^2 + \alpha + 2 = 0$. Express

$$(\alpha^2+\alpha+1)(\alpha^2-\alpha) \qquad \text{and} \qquad (\alpha-1)^{-1}$$

   in the form $a\alpha^2 + b\alpha + c$ with $a, b, c \in \mathbb{Q}$.

3. Let

$$A = \begin{pmatrix} 1-x & 1+x & x \\ x & 1-x & 1 \\ 1+x & 2x & 1 \end{pmatrix} \in M_{3\times 3}(\mathbb{Q}[x]).$$

   Determine the $\mathbb{Q}[x]$-module $V$ presented by $A$. Is $V$ a cyclic $\mathbb{Q}[x]$-module?

4. (a) Compute the characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}.$$

   (b) What is the characteristic polynomial of a matrix in rational canonical form?

   (c) Use (b) to prove the Cayley-Hamilton Theorem: If $A$ is a square matrix and $p(t)$ is its characteristic polynomial, then $p(A) = 0$.

5. Let $R = \mathbb{Q}[x]$ and suppose that the $R$-module $M$ is a direct sum of four cyclic modules

$$\mathbb{Q}[x]/((x-1)^3) \oplus \mathbb{Q}[x]/((x^2+1)^2) \oplus \mathbb{Q}[x]/((x-1)(x^2+1)^4) \oplus \mathbb{Q}[x]/((x+2)(x^2+1)^2).$$

   (a) Decompose $M$ into a direct sum of cyclic modules of the form $\mathbb{Q}[x]/(f_i^{m_i})$, where $f_i$ are monic irreducible polynomials in $\mathbb{Q}[x]$ and $m_i > 0$.

   (b) Find monic $d_1, d_2, \ldots, d_k \in \mathbb{Q}[x]$ with positive degree such that

      (i) if $i \in \{1, \ldots, k-1\}$ then $d_i | d_{i+1}$, and

      (ii) $M \cong \mathbb{Q}[x]/(d_1) \oplus \cdots \oplus \mathbb{Q}[x]/(d_k)$.

   (c) Identify the $\mathbb{Q}[x]$-module $M$ with the vector space $M$ over $\mathbb{Q}$ together with a linear operator

$$\begin{array}{cccc} X\colon & M & \to & M \\ & v & \mapsto & xv. \end{array}$$

   Suppose the matrix of $X$ is $A$ with respect to a $\mathbb{Q}$-vector space basis of $M$. Determine the minimal and characteristic polynomials of $A$ and the dimension of $M$ over $\mathbb{Q}$.

6. Let $\lambda \in \mathbb{C}$ and $m \in \mathbb{Z}_{>0}$. Let $V$ be the cyclic $\mathbb{C}[t]$-module

$$V = \frac{\mathbb{C}[t]}{((t-\lambda)^m)}.$$

 (a) Show that

$$\{w_0 = 1, w_1 = t-\lambda, w_2 = (t-\lambda)^2, \ldots, w_{m-1} = (t-\lambda)^{m-1}\}$$

 is a basis of $V$ as $\mathbb{C}$-vector space.

 (b) Show that, with respect to the basis in (a), the matrix of

$$\begin{array}{rccc} T\colon & V & \to & V \\ & v & \mapsto & tv \end{array} \quad \text{is of the form} \quad A = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda \end{pmatrix} \in M_{m\times m}(\mathbb{C}).$$

7. Suppose that $V$ is an 8 dimensional complex vector space and $T : V \to V$ is a linear operator. Using $T$ we make $V$ into a $\mathbb{C}[t]$-module in the usual way. Suppose that as a $\mathbb{C}[t]$-module

$$V \cong \frac{\mathbb{C}[t]}{((t+5)^2)} \oplus \frac{\mathbb{C}[t]}{((t-3)^3(t+5)^3)}.$$

What is the Jordan (normal) form for the transformation $T$? What are the minimal and characteristic polynomials of $T$?

8. Let $V$ be an $F[t]$-module and $(v_1, \ldots, v_n)$ a basis of $V$ as an $F$-vector space. Let $T : V \to V$ be a linear operator and $A \in M_{n\times n}(F)$ the matrix of $T$ with respect to the basis $(v_1, \ldots, v_n)$. Prove that the $F[t]$-matrix $tI - A$ is a presentation matrix of $(V, T)$ regarded as a $F[t]$-module.

9. Determine the Jordan normal form of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in M_{3\times 3}(\mathbb{C})$$

by decomposing the $\mathbb{C}[t]$-module $V$ presented by the matrix $tI - A \in M_{3\times 3}(\mathbb{C}[t])$.

10. Find all possible Jordan normal forms for a matrix $A \in M_{5\times 5}(\mathbb{C})$ whose characteristic polynomial is $(t+2)^2(t-5)^3$.