19.6.1 $\mathbb{Z}$-Modules

1. Show that if $p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ is the prime decomposition of the integer $n \geq 2$ then
$$\frac{\mathbb{Z}}{n\mathbb{Z}} \cong \frac{\mathbb{Z}}{p_1^{a_1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k^{a_k}\mathbb{Z}}.$$
2. Let $p \in \mathbb{Z}_{>0}$ be prime. Show that $\mathbb{Z}/p^2\mathbb{Z}$ is not isomorphic to the direct sum of two cyclic groups.
3. Let $p \in \mathbb{Z}_{>0}$ prime and let $n \in \mathbb{Z}_{\geq 0}$. Show that the $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z}$ is not a direct sum of two nontrivial $\mathbb{Z}$-modules.
4. Show that the $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z}$, where p is a prime and n a positive integer, is not a direct sum of two non-zero $\mathbb{Z}$-modules.
5. Show (without using the Structure Theorem) that $\mathbb{Z}_{p^r}$ is not a direct sum of two abelian groups when p is a prime and r is a positive integer.
6. Up to isomorphism, how many abelian groups of order 96 are there?
7. How many abelian groups of order 136 are there? Give the primary and invariant factor decompositions of each.
8. Determine which if the following abelian groups are isomorphic:
$$C_6 \oplus C_{50} \oplus C_{60}, \quad C_{20} \oplus C_{30} \oplus C_{30}, \quad C_{12} \oplus C_{25} \oplus C_4 \oplus C_{15}, \quad C_{30} \oplus C_{50} \oplus C_{12}.$$
9. Determine which of the following abelian groups are isomorphic:
$$C_6 \oplus C_{100} \oplus C_{15}, \qquad C_{50} \oplus C_6 \oplus C_{30}, \qquad C_{30} \oplus C_{300}, \qquad C_{60} \oplus C_{150}.$$
10. Determine the invariant factors of the abelaingroup $C_{100} \oplus C_{36} \oplus C_{150}$.
11. Determine, up to isomorphism, all abelian groups of order 1080.
12. Determine, up to isomorphism, all abelian groups of order 360.
13. Give a list of all the different abelian groups of order 54.
14. List, up to isomorphism, all abelian groups of order 360. Give the primary decomposition and the annihilator (as a $\mathbb{Z}$-module) of each group.
15. If an abelian group has torsion invariants (or equivalently invariant factors) $2, 6, 54$ determine its primary decomposition.
16. Determine all abelian groups of order 72.
17. List, up to isomorphism, all abelian groups of order 504. Give the primary decomposition and annihilator (as a $\mathbb{Z}$-module) of each group.
18. Give the primary decomposition and invariant factor decomposition of the $\mathbb{Z}$-module $\mathbb{Z}/20\mathbb{Z} \oplus \mathbb{Z}/40\mathbb{Z} \oplus \mathbb{Z}/100\mathbb{Z}$.

19. Determine which of the following abelian groups are isomorphic:

$$C_{12} \oplus C_{50} \oplus C_{30}, \quad C_4 \oplus C_{12} \oplus C_{15} \oplus C_{25}, \quad C_{20} \oplus C_{30} \oplus C_{30}, \quad C_6 \oplus C_{50} \oplus C_{60}.$$

20. Show that $\mathbb{Q}$ is torsion-free but not free as a $\mathbb{Z}$-module.

21. Show that $\mathbb{Q}$ is not free as a $\mathbb{Z}$-module (remember the basis may be infinite).

22. Show that a finitely generated torsion free module over a Principal Ideal Domain R is a free R-module.

23. Using the Structure theorem or otherwise show that a fintiely generate torsion-free abelian group is a free abelian group.

24. Let N be the submodule of the $\mathbb{Z}$-module $\mathbb{Z}^3$ generated by the vectors

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 9 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 12 \\ 3 \end{pmatrix}.$$

Find a basis $\{b_1, b_2, b_3\}$ of $\mathbb{Z}^3$, and $d_1, d_2, d_3 \in \mathbb{Z}$, such that the nonzero elements in the set $\{d_1 b_1, d_2 b_2, d_3 b_3\}$ form a basis for N.

25. Let $N = \{(x, y, z) \in \mathbb{Z}^3 \mid x + y + z = 0\}$. Is N a free $\mathbb{Z}$-module? If so, find a basis.

26. Let $R = \mathbb{Z}$ and $F = \mathbb{Z}^3$. Let $N = \{(x, y, z) \in F \mid x + y + z = 0\}$. Show that N is a submodule of F and find a basis of N.

27. Explain why there is no $u \in \mathbb{Z}^3$ such that $\{(2, 4, 1), (2, -1, 1), u\}$ is a basis of $\mathbb{Z}^3$.

28. Let A be a $m \times n$ matrix. Then A determines a $\mathbb{Z}$-module homomorphism

$$f_A : \mathbb{Z}^n \to \mathbb{Z}^m$$

by $f_A(\mathbf{v}) = A\mathbf{v}$, with elements of $\mathbb{Z}^n$ and $\mathbb{Z}^m$ written as column vectors.

Similarly, the transpose A^T of A determines a $\mathbb{Z}$-module homomorphism

$$f_{A^T} : \mathbb{Z}^m \to \mathbb{Z}^n.$$

Prove the cokernels of f_A and f_{A^T} have isomorphic torsion subgroups. [Recall the cokernel of $\phi : M \to N$ is defined as the quotient $N/\mathrm{im}(\phi)$.]

29. Let $A \in M_{d\times d}(\mathbb{Z})$ and let φ be the $\mathbb{Z}$-module morphism given by

$$\begin{array}{rccc} \varphi: & \mathbb{Z}^k & \to & \mathbb{Z}^k \\ & v & \mapsto & Av. \end{array} \qquad \text{Show that} \quad \mathrm{Card}\left(\frac{\mathbb{Z}^k}{\mathrm{im}(\varphi)}\right) = \begin{cases} |\det(A)|, & \text{if } \det(A) \neq 0, \\ \infty, & \text{if } \det(A) = 0. \end{cases}$$

30. Let

$$A = \begin{pmatrix} 3 & 8 & 7 & 9 \\ 2 & 4 & 6 & 6 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

(a) Find the Smith Normal form (over $\mathbb{Z}$) of the matrix A.

(b) If M is a $\mathbb{Z}$-module with presentation matrix A, then show that $M \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

(c) If N is a $\mathbb{Q}$-module with presentation matrix A, identify N.

31. Find an isomorphic direct sum of cyclic groups, where V is an abelian group generated by x, y, z and subject to relations:

    (a) $3x + 2y + 8z = 0$, $2x + 4z = 0$

    (b) $x + y = 0$, $2x = 0$, $4x + 2z = 0$, $4x + 2y + 2z = 0$

    (c) $2x + y = 0$, $x - y + 3z = 0$

    (d) $4x + y + 2z = 0, 5x + 2y + z = 0, 6y - 6z = 0$.

32. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and let $N = \mathbb{Z}\text{-span}\{(0, 3)\}$.
    Write M/N as a direct sum of cyclic submodules.

33. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and let $N = \mathbb{Z}\text{-span}\{(2, 0), (0, 3)\}$.
    Write M/N as a direct sum of cyclic submodules.

34. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and let $N = \mathbb{Z}\text{-span}\{(2, , 3)\}$.
    Write M/N as a direct sum of cyclic submodules.

35. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and let $N = \mathbb{Z}\text{-span}\{(6, 9)\}$.
    Write M/N as a direct sum of cyclic submodules.

36. Find a direct sum of cyclic groups which is isomorphic to the abeliangroup $\mathbb{Z}^3/N$, where N is generated by $\{(2, 2, 2), (2, 2, 0), (2, 0, 2)\}$.

37. Find an isomorphic direct product of cyclic groups and the invariant factors of V, where V is an abeliangroup

$$\text{generated by } x, y, z \quad \text{with relations} \quad 3x + 2y + 8z = 0 \text{ and } 2x + 4z = 0.$$

38. Find an isomorphic direct product of cyclic groups and the invariant factors of V, where V is an abeliangroup

$$\text{generated by } x, y, z \quad \text{with relations} \quad x + y = 0,\ 2x = 0,\ 4x + 2z = 0 \text{ and } 4x + 2y + 2z = 0.$$

39. Find an isomorphic direct product of cyclic groups and the invariant factors of V, where V is an abeliangroup

$$\text{generated by } x, y, z \quad \text{with relations} \quad 2x + y = 0 \text{ and } x - y + 3z = 0.$$

40. Find an isomorphic direct product of cyclic groups and the invariant factors of V, where V is an abeliangroup

$$\text{generated by } x, y, z \quad \text{with relations} \quad 4x + y + 2z = 0,\ 5x + 2y + z = 0 \text{ and } 6 - 6z = 0.$$

41. Let $\varphi\colon \mathbb{Z}^4 \to \mathbb{Z}^3$ be the $\mathbb{Z}$-module homomorphism determined by

$$\varphi(1, 0, 0, 0) = (14, 8, 2), \quad \varphi(0, 1, 0, 0) = (12, 6, 0), \quad \varphi(0, 0, 1, 0) = (18, 12, 0), \quad \varphi(0, 0, 0, 1) = (0, 6, 0).$$

    (a) Find bases for $\mathbb{Z}^3$ and $\mathbb{Z}^4$ such that the matrix of φ with respect to these bases is in Smith normal form.

(b) Find the invariant factor decomposition of the $\mathbb{Z}$-module $\mathbb{Z}^4/\mathrm{im}(\varphi)$.

42. Let A be the abelian group with generators a, b, c and relations

$$3a = b - c, \qquad 6a = 2c, \qquad 3b = 4c.$$

(a) Find the generator mmatrix for A.

(b) Bring this matrix into Smith normal form, carefully recording each step.

(c) By the classification theorem for finitely generated abelain groups, A is isomorphic to a cartesian product of cyclic groups of prime power order and/or copies of $\mathbb{Z}$. Which group is it?

(d) Write down an explicit isomorphism from A to the group you have identified in part (c).

(e) Interpret the meaning of the three matrices L, D and R in this context.

43. (a) Let $N \subseteq \mathbb{Z}^3$ be the submodule generated by the set $\{(2,4,1),(2,-1,1)\}$. Find a basis $\{f_1, f_2, f_3\}$ for $\mathbb{Z}^3$, and elements $d_1, d_2, d_3 \in \mathbb{Z}$ such that the nonzero elements $\{d_1f_1, d_2f_2, d_3f_3\}$ form a basis for N and $d_1|d_2|d_3$.

(b) Write $\mathbb{Z}^3/N$ as a direct sum of nontrivial cyclic $\mathbb{Z}$-modules.

44. Let M be the $\mathbb{Z}$-module given by $M = \mathbb{Z}^4/N$, where N is the submodule of $\mathbb{Z}^4$ generated by $\{(15,1,8,1),(0,2,0,2),(7,1,4,1)\}$.

(i) Write M as a direct sum of non-trivial cyclic $\mathbb{Z}$-modules.

(ii) What is the torsion-free rank of M?

45. Determine the invariant factor decomposition of the abelian group given by $\mathbb{Z}^3/N$ where N is the submodule of $\mathbb{Z}^3$ generated by $\{(7,4,1),(8,5,2),(7,4,1,5)\}$.

46. Let A be the abelian group given by $A = \mathbb{Z}^3/N$ where N is the submodule of $\mathbb{Z}^3$ generated by $\{(-4,2,6),(-6,2,6),(7,4,15)\}$. If

$$R = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}, \qquad X = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 2 \\ 21 & 3 & 7 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 3 & -1 \\ 1 & -2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$$

then

$$X^{-1}RY = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

(i) Find a basis $\{b_1, b_2, b_3\}$ of $\mathbb{Z}^3$ such that $\{d_1b_1, d_2b_2, d_3b_3\}$ generates N.

(ii) Find elements $u \in A$ and $v \in A$ that generate A and are such that $2u = 0$ and $6v = 0$.

47. Determine the torsion-free rank and the torsion invariants of the abelian group given by generators a, b, c and relations

$$7a + 4b + c = 8a + 5b + 2c = 9a + 6b + 3c = 0.$$

48. Deteremine the torsion free rank and the torsion invariants of the abelian group presented by generators a, b, c and relations

$$7a + 4b + c = 8a + 5b + 2c = 9a + 6b + 3c = 0.$$

49. Let F be the $\mathbb{Z}$-module $\mathbb{Z}^3$ and let N be the submodule generated by
$$\{(4,-4,4),(-4,4,8),(16,20,4)\}.$$
Calculate the invariant factor decomposition of F/N.

50. Let F be the $\mathbb{Z}$-module $\mathbb{Z}^3$ and let N be the submodule generated by
$$\{(4,-4,4),(-4,4,8),(16,20,4)\}.$$
Calculate the primary decomposition of F/N.

51. Let F be the $\mathbb{Z}$-module $\mathbb{Z}^3$ and let N be the submodule generated by
$$\{(4,-4,4),(-4,4,8),(16,20,4)\}.$$
Find a basis $\{f_1,f_2,f_3\}$ for F and integers d_1,d_2,d_3 such that $d_1|d_2|d_3$ and $\{d_1f_1,d_2f_2,d_3f_3\}$ is a basis for N.

52. Determine the invariant factors and the torsion free rank of the abelian group M generated by x,y,z subject to the relations
$$2x-4y+2z=0 \qquad \text{and} \qquad -2x+10y+4z=0.$$

53. Let M be the abelian group generated by x,y,z subject to the relations
$$2x-4y+2z=0 \qquad \text{and} \qquad -2x+10y+4z=0.$$
Express M as a direct sum of cyclic groups in a unique way.

54. Determine the invariant factors and the torsion free rank of the abelian group M generated by x,y,z subject to the relations
$$9x+12y+6z=0 \qquad \text{and} \qquad 6x+3y-6z=0.$$

55. Let M be the abelian group generated by x,y,z subject to the relations
$$9x+12y+6z=0 \qquad \text{and} \qquad 6x+3y-6z=0.$$
Express M as a direct sum of cyclic groups in a unique way and find the primary components of M.

56. Determine the invariant factors of the abelian group generated by a,b,c,d subject to the relations $3a-3c=6b+3c-6d=3b+2c-3d=3a+6c+3d=0$.

57. Consider the linear transformation α acting on $\mathbb{Z}^3$ given by
$$\alpha(e_1)=e_2+e_3, \qquad \alpha(e_2)=2e_2, \quad \alpha(e_3)=e_1+2e_2.$$
Show that the minimal polynomial and the characteristic polynomial for α are the same.

58. Consider the linear transformation α acting on $\mathbb{F}_3^3$ given by
$$\alpha(e_1)=e_2+e_3, \qquad \alpha(e_2)=2e_2, \quad \alpha(e_3)=e_1+2e_2.$$
Determine the Jordan normal form of α.

59. Let A be the abelain group generated by a, b, c with relations

$$7a + 4b + c = 8a + 5b + 2c = 9a + 6b + 3c = 0.$$

Express this group as a direct sum of cyclic groups.

60. Let N be the submodules of the $\mathbb{Z}$-module $F = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ generated by

$$\{(1, 0, -1), (4, 3, -1), (0, 9, 3), (3, 12, 3)\}.$$

(i) Find a basis $\{b_1, b_2, b_3\}$ of F and $d_1, d_2, d_3 \in \mathbb{Z}$ such that the non-zero elements of the set $\{d_1b_1, d_2b_2, d_3b_3\}$ form a basis for N (as a $\mathbb{Z}$-module).

(ii) Give a direct sum of non-trivial cyclic groups that is isomorphic to F/N.

61. Let A be an abelian group with presentation

$$\langle a, b, c \mid 2a + b = 0, 3a + 3c = 0\rangle.$$

Give the primary decomposition of A. What is the torsion free rank of A?

62. Let F be the $\mathbb{Z}$-module $F = \mathbb{Z}^4$ and let N be the submodule of F generated by

$$\{(1, 1, 1, 1), (1, -1, 1, -1), (1, 3, 1, 3)\}.$$

Give a direct sum of non-trivial cyclic $\mathbb{Z}$-modules that is isomorphic to F/N.

63. Let $\mathcal{S} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ be the standard basis of the free $\mathbb{Z}$-module $\mathbb{Z}^3$ and let N be the submodule with basis

$$\mathcal{B} = \{(2, 2, 2), (0, 8, 4)\}.$$

Find a new basis $\{f_1, f_2, f_3\}$ for $\mathbb{Z}^3$ and elements $d_1, d_2, d_3 \in \mathbb{Z}$ such that the non-zero elements of the set $\{d_1f_1, d_2f_2, d_3f_3\}$ form a basis for N and $d_1|d_2|d_3$.

64. Find the invariant factor matrix over $\mathbb{Z}$ that is equivalent to the matrix

$$\begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}$$

65. Consider the abelian group A with generators x, y, z subject to the defining relations $7x + 5y + 2z = 0$, $3x + 3y = 0$ and $13x + 11y + 2z = 0$. Find a direct sum of cyclic groups which is isomorphic to A. Explain in sense your answer is unique.

66. Let M be the $\mathbb{Z}[i]$-module given by $M = \mathbb{Z}[i]^3/N$ where N is the submodule of $\mathbb{Z}[i]^3$ generated by

$$\{(-i, 0, 0), (1 - 2i, 0, 1 + i), (1 + 2i, -2i, 1 + 3i)\}.$$

If

$$A = \begin{pmatrix} -i & 1-2i & 1+2i \\ 0 & 0 & -2i \\ 0 & 1+i & 1+3i \end{pmatrix}, \qquad X = \begin{pmatrix} i & 2 & 0 \\ 0 & 0 & -i \\ 0 & i & 1+i \end{pmatrix}, \qquad Y = \begin{pmatrix} -i & i & 1+i \\ 0 & 1 & -i \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$X^{-1}AY = \begin{pmatrix} i & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) Write M as a direct sum of nontrivial cyclic $\mathbb{Z}[i]$-modules.

(b) Calculate the annihilator of M.

(c) Find an element $u \in M - \{0\}$ with the property that $iu = u$.

67. Let V be the $\mathbb{Z}[i]$-module $(\mathbb{Z}[i])^2/N$ where

$$N = \mathrm{span}_{\mathbb{Z}[i]}\{(1+i, 2-i), (3, 5i)\}.$$

Write V as a direct sum of cyclic modules.

68. Let V be the $\mathbb{Z}[i]$-module $(\mathbb{Z}[i])^2/N$, where $N = \mathbb{Z}[i]\text{-span}\{(1+i, 2-i), (3, 5i)\}$. Write V as a direct sum of cyclic modules.