04.12.2024 ① L10 W4L

Crystals

For $i \in \{1, \dots, n\}$ let $\varepsilon_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) \in \mathbb{Z}^n$.

Let

$$B(\square) = \{ \boxed{1} \xrightarrow{\tilde{f}_1} \boxed{2} \xrightarrow{\tilde{f}_2} \cdots \xrightarrow{\tilde{f}_{n-1}} \boxed{n} \} \quad \text{with } wt(\boxed{i}) = \varepsilon_i$$

A <u>crystal</u> is an element B of the category generated by $B(\square)$ under direct sums and tensor products.

A crystal is a set B with functions

$$wt : B \to \mathbb{Z}^n \quad \text{and} \quad \tilde{f}_i : B \to B \sqcup \{0\}$$

for $i \in \{1, \dots, n-1\}$.

The <u>crystal graph</u> of B is the labeled graph with

Vertices: B Labeled edges: $b \xrightarrow{\tilde{f}_i} \tilde{f}_i b$

A <u>crystal morphism</u> from B_1 to B_2 is a function $\Phi : B_1 \to B_2$ such that

$$wt(\Phi(b)) = wt(b) \quad \text{and}$$
$$\tilde{f}_i(\Phi(b)) = \Phi(\tilde{f}_i b)$$

for $b \in B_1$ and $i \in \{1, \dots, n-1\}$.

The <u>character</u> of a crystal B is

$$\mathrm{char}(B) = \sum_p x^{wt(p)}$$

where $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$ if $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$.

The <u>direct sum</u> of crystals B_1 and B_2 is

$$B_1 \oplus B_2 = B_1 \sqcup B_2 \text{ with}$$

wt and $\tilde{f}_i$ inherited from $B_1 \hookrightarrow B$ and $B_2 \hookrightarrow B$.

For $i \in \{1, \ldots, n-1\}$ define

$$\tilde{e}_i : B \longrightarrow B \cup \{0\} \quad \text{by} \quad \tilde{e}_i(\tilde{f}_i b) = b$$

if $\tilde{f}_i b \neq 0$ and

$\tilde{e}_i(b) = 0$ if there does not exist b' such that $b = \tilde{f}_i b'$.

For $b \in B$ let $d_i^+(b)$ and $d_i^-(b)$ be given by

$$\tilde{e}_i^{\,d_i^+(b)}(b) \neq 0 \quad \text{and} \quad \tilde{e}_i^{\,d_i^+(b)+1}(b) = 0, \quad \text{and}$$

$$\tilde{f}_i^{\,d_i^-(b)}(b) \neq 0 \quad \text{and} \quad \tilde{f}_i^{\,d_i^-(b)+1}(b) = 0.$$

Then

$$\tilde{e}_i^{\,d_i^+(b)} b \xrightarrow{\tilde{f}_i} \cdots \xrightarrow{\tilde{f}_i} \tilde{e}_i b \xrightarrow{\tilde{f}_i} b \xrightarrow{\tilde{f}_i} \tilde{f}_i b \xrightarrow{\tilde{f}_i} \cdots \xrightarrow{\tilde{f}_i} \tilde{f}_i^{\,d_i^-(b)} b$$

is the <u>i-string</u> of b.

The <u>tensor product</u> of crystals B_1 and B_2 is

$$B_1 \otimes B_2 = B_1 \times B_2 = \{ b_1 \otimes b_2 \mid b_1 \in B_1,\ b_2 \in B_2 \}$$

with

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2)$$

and

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } d_i^+(b_1) > d_i^-(b_2), \\ b_1 \otimes \tilde{f}_i b_2, & \text{if } d_i^+(b_1) \leq d_i^-(b_2). \end{cases}$$

Then

$$char(B_1 \oplus B_2) = char(B_1) + char(B_2)$$
$$char(B_1 \otimes B_2) = char(B_1)\, char(B_2)$$

and

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } d_i^+(b_1) \geq d_i^-(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } d_i^+(b) < d_i^-(b_2) \end{cases}$$

<u>HW</u>: Show that if B_1, B_2, B_3 are crystals then

$$\Phi : (B_1 \otimes B_2) \otimes B_3 \longrightarrow B_1 \otimes (B_2 \otimes B_3)$$
$$b_1 \otimes b_2 \otimes b_3 \longmapsto b_1 \otimes b_2 \otimes b_3$$

is a crystal isomorphism.

A <u>highest weight element</u> of a crystal B is $b \in B$ such that

if $i \in \{1, \dots, n-1\}$ then $\tilde{e}_i b = 0$.

Let

$B^+ = \{\text{highest weight elements of } B\}$

$B^+_\lambda = \{b \in B^+ \mid wt(b) = \lambda\}$.

A crystal B is <u>irreducible</u>, or <u>simple</u>, if B has ~~not~~ ~~proper~~ subcrystals except $\emptyset$ and B.

A <u>subcrystal</u> of B is a subset of B closed under the operators $\tilde{e}_i$ and $\tilde{f}_i$.

<u>Theorem</u>

(a) A crystal B is irreducible if and only if the crystal graph of B is connected.

(b) A crystal B is irreducible if and only if $\mathrm{Card}(B^+) = 1$.

__Proposition__ Assume B_1 and B_2 are irreducible crystals.

(a) If $\Phi : B_1 \to B_2$ is a crystal morphism then Φ is a crystal isomorphism, $B_1 \simeq B_2$

(b) If $B_1^+ = \{b_1^+\}$ and $B_2^+ = \{b_2^+\}$ then

$$B_1 \simeq B_2 \text{ if and only if } wt(b_1^+) = wt(b_2^+)$$

__Theorem__ Two crystals are isomorphic if and only if

$$char(B_1) = char(B_2)$$

__Proof__ Decompose B_1 and B_2 into connected components. Let

$B(\lambda)$ be the irreducible crystal of highest weight λ; $B(\lambda)^+ = \{b_\lambda^+\}$ and $wt(b_\lambda^+) = \lambda$.

Then

$$B_1 \simeq \bigsqcup_{p \in B_1^+} B(wt(p)) \simeq B_2$$

So

$$char(B_1) = \sum_{p \in B_1^+} char(B(wt(p))) = char(B_2).$$