13.11.2014
ADM ①
L7
W3L1

Fundamental symmetric functions

Define

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}) \quad \text{and}$$

$$(a;q)_\infty = (1-a)(1-aq)(1-q^2a)\cdots$$

Define

$$g_r = g_r(x;q,t)$$
$$q_r = q_r(x;t)$$
$$h_r = h_r(x)$$
$$e_r = e_r(x)$$

by the generating functions

$$\sum_{r\in\mathbb{Z}_{\geq 0}} g_r z^r = \prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(x_iz;q)_\infty} \qquad g_r = \frac{(t;q)_r}{(q;q)_r} P_{(r)}(x;q,t)$$

one row Macdonald polys

$$\sum_{r\in\mathbb{Z}_{\geq 0}} q_r z^r = \prod_{i=1}^n \frac{1-tx_iz}{1-x_iz} \qquad q_r = (1-t)P_{(r)}(x;0,t)$$

one row Hall-Littlewood polys

$$\sum_{r\in\mathbb{Z}_{\geq 0}} h_r z^r = \prod_{i=1}^n \frac{1}{1-x_iz} \qquad h_r = s_{(r)}(x)$$

one row Schur functions

$$\sum_{r\in\mathbb{Z}_{\geq 0}} e_r z^r = \prod_{i=1}^n (1+x_iz) \qquad e_r = s_{(1^r)}(x) = P_{(1^r)}(x;0,t) = P_{(1^r)}(x;q,t)$$

one column Schur functions

"Extensions" 13.11.2024

ADM ②

L7

Define $\tilde{g}_r = \tilde{g}_r(x;q,t,u)$ and $\hat{q}_r = \hat{q}_r(x;t,u)$ by

$$\sum_{r\in\mathbb{Z}_{\geq 0}} \tilde{g}_r z^r = \prod_{i=1}^{n} \frac{(tx_iz;q)_\infty}{(ux_iz;q)_\infty}$$

$$\sum_{r\in\mathbb{Z}_{\geq 0}} \hat{q}_r z^r = \prod_{i=1}^{n} \frac{(1-tx_iz)}{(1-ux_iz)}$$

Then

$$\hat{q}_r(x;t,u) = \tilde{g}_r(x;0,t,u)$$

$$h_r(x) = \tilde{g}_r(x;0,0,1)$$

$$q_r(x;t) = \tilde{g}_r(x;0,t,1)$$

$$e_r(x) = \tilde{g}_r(x;0,-1,0)$$

Since

$$g_r(x;q,t) = \tilde{g}_r(x;q,t,1) \quad \text{and}$$

$$\tilde{g}_r(x_1,\dots,x_n;q,t,u) = u^r g_r(u^{-1}x_1,\dots,u^{-1}x_n;q,t,u)$$
$$= u^r g_r(x;q,tu^{-1})$$

then any formula for $\tilde{g}_r$ immediately converts to a formula for g_r and vice versa.

Formulas in terms of sequences

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$$\frac{1}{1-ux_iz} = 1 + ux_iz + u^2x_i^2z^2 + \cdots$$

$$\frac{1-tx_iz}{1-ux_iz} = \cancel{(u-t)(1+ux_iz+u^2x_i^2z^2+\cdots)}$$

$$= 1 + \frac{(u-t)x_iz}{1-ux_iz}$$

$$= 1 + (u-t)x_iz(1+ux_iz+u^2x_i^2z^2+\cdots)$$

Then

$$\prod_{i=1}^{n} \frac{1-tx_iz}{1-ux_iz} = \left(\frac{1-tx_1z}{1-ux_1z}\right)\cdots\left(\frac{1-tx_nz}{1-ux_nz}\right)$$

gives

$$\hat{e}_r = \sum_{1\le i_1\le\cdots\le i_r\le n} (u-t)^{1+\#\{j \mid i_j<i_{j+1}\}} u^{\#\{j \mid i_j=i_{j+1}\}} x_{i_1}x_{i_2}\cdots x_{i_r}$$

Then

$$\left.\frac{1}{u-t}\hat{e}_r\right]_{t=u} = p_r = \sum_{i_1=i_2=\cdots=i_r} x_{i_1}\cdots x_{i_r}.$$

Up-down sequences

Using

$$\prod_{i=1}^{n}\frac{1-tx_iz}{1-ux_iz} = \left(\frac{1}{1-ux_1z}\right)\cdots\left(\frac{1}{1-ux_nz}\right)(1-tx_1z)\cdots(1-tx_nz)$$

gives

$$\hat{q}_r = \sum_{i_1 \le i_2 \le \cdots \le i_k > i_{k+1} > \cdots > i_r} u^{k-1} (-t)^{r-k} x_{i_1} \cdots x_{i_k} x_{i_{k+1}} \cdots x_{i_r}$$

So

$$e_r \stackrel{?}{=} \hat{q}_r(-1 \qquad) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}$$

and

$$h_r = \hat{q}_r(x; 0, 1) = \sum_{i_1 \le \cdots \le i_r} x_{i_1} \cdots x_{i_r}.$$

Monomial symmetric functions

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$ the monomial symmetric functions is

$$m_\lambda = \sum_{\gamma \in S_n \lambda} x^\gamma, \quad \text{where } x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$$

The q-binomial theorem gives

$$\frac{(tx_i z; q)_\infty}{(x_i z; q)_\infty} = \sum_{r \in \mathbb{Z}_{\ge 0}} \frac{(t; q)_r}{(q; q)_r} x_i^r z^r.$$

Using the q-binomial theorem on the product

$$\prod_{i=1}^{n} \frac{(tx_i z;q)_\infty}{(x_i z;q)_\infty} = \frac{(tx_1 z;q)_\infty}{(x_1 z;q)_\infty} \cdots \frac{(tx_n z;q)_\infty}{(x_n z;q)_\infty}$$

gives

$$\hat{g}_r = \sum_{|\mu|=r} u^r \frac{(tu^{-1};q)_\mu}{(q;q)_\mu} m_\mu$$

where

$$\frac{(tu^{-1};q)_\mu}{(q;q)_\mu} = \frac{(tu^{-1};q)_{\mu_1} \cdots (tu^{-1};q)_{\mu_\ell}}{(q;q)_{\mu_1} \cdots (q;q)_{\mu_\ell}}$$

if $\mu = (\mu_1, \dots, \mu_\ell)$. So

$$\hat{q}_r = \sum_{|\mu|=r} u^{r-\ell(\mu)} (u-t)^{\ell(\mu)} m_\mu$$

$$g_r = \sum_{|\mu|=r} \frac{(t;q)_\mu}{(q;q)_\mu} m_\mu$$

$$q_r = \sum_{|\mu|=r} (1-t)^{\ell(\mu)} m_\mu$$

$$h_r = \sum_{|\mu|=r} m_\mu, \qquad e_r = m_{(1^r)}, \qquad p_r = m_{(r)}.$$