Power sums and the Cauchy-Macdonald kernel

The *power sums* $p_r \in \mathbb{C}[x_1,\dots,x_n]$ are

$$p_0 = 1 \quad \text{and} \quad p_r = x_1^r + \cdots + x_n^r \quad \text{for } r \in \mathbb{Z}_{>0}$$

For $\lambda = (\lambda_1,\dots,\lambda_\ell) \in \mathbb{Z}_{\geq 0}^n$ define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}$$

Since

$$\log(1-z) = \int \frac{-1}{1-z}\,dz$$

$$= \int -(1+z+z^2+\cdots)\,dz$$

$$= -z - \tfrac{1}{2}z^2 - \tfrac{1}{3}z^3 - \cdots = -\sum_{r \in \mathbb{Z}_{>0}} \tfrac{1}{r} z^r$$

Then

$$\log\left(\prod_{i=1}^n \frac{(tx_i z;q)_\infty}{(ux_i z;q)_\infty}\right) = \sum_{i=1}^n \sum_{\ell \in \mathbb{Z}_{\geq 0}} \log(1-tx_i z q^\ell) - \log(1-ux_i z q^\ell)$$

$$= \sum_{i=1}^n \sum_{\ell \in \mathbb{Z}_{\geq 0}} \sum_{r \in \mathbb{Z}_{>0}} \left(-\tfrac{1}{r} t^r x_i^r q^{\ell r} z^r + \tfrac{1}{r} u^r x_i^r q^{\ell r} z^r\right)$$

$$= \sum_{\ell \in \mathbb{Z}_{\geq 0}} \sum_{r \in \mathbb{Z}_{>0}} \tfrac{1}{r}(u^r - t^r)\, q^{\ell r} p_r z^r$$

$$= \sum_{r \in \mathbb{Z}_{>0}} \left(\frac{u^r - t^r}{1-q^r}\right) \frac{p_r}{r} z^r. \qquad (*)$$

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Define

$$z_\lambda(q,t,u) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{u^{\lambda_i}-t^{\lambda_i}}$$

where

$$z_\lambda = 1^{m_1} m_1! \, 2^{m_2} m_2! \cdots \quad \text{for } \lambda = (1^{m_1} 2^{m_2} \dots)$$

Taking the exponential of both sides of $(*)$ gives

$$\prod_{i=1}^{n} \frac{(tx_i z;q)_\infty}{(ux_i z;q)_\infty} = \sum_{r\in\mathbb{Z}_{\geq 0}} \Big(\sum_{|\lambda|=r} \frac{1}{z_\lambda(q,t,u)} P_\lambda(x) \Big) z^r$$

So

$$\hat{g}_r = \sum_{|\lambda|=r} \frac{1}{z_\lambda(q,t,u)} P_\lambda(x) = \sum_{|\lambda|=r} \Big(\prod_{i=1}^{\ell(\lambda)} \Big(\frac{u^{\lambda_i}-t^{\lambda_i}}{1-q^{\lambda_i}} \Big) \Big) \frac{P_\lambda}{z_\lambda}$$

Then

$$\hat{q}_r = \hat{q}_r(x;t,u) = \hat{g}_r(x;0,t,u) = \sum_{|\lambda|=r} \Big(\prod_{i=1}^{\ell(\lambda)} (u^{\lambda_i}-t^{\lambda_i}) \Big) \frac{P_\lambda}{z_\lambda}$$

$$q_r = q_r(x;t) = \hat{q}_r(x;t,1) = \sum_{|\lambda|=r} \Big(\prod_{i=1}^{\ell(\lambda)} (1-t^{\lambda_i}) \Big) \frac{P_\lambda}{z_\lambda}$$

$$h_r = \sum_{|\lambda|=r} \frac{1}{z_\lambda} P_\lambda(x) \quad \text{and} \quad e_r = \sum_{|\lambda|} (-1)^{r-\ell(\lambda)} \frac{P_\lambda}{z_\lambda}$$

The Cauchy-Macdonald kernel

For $\lambda=(\lambda_1,\dots,\lambda_\ell)\in\mathbb{Z}^n_{\geq 0}$ define

$$\tilde{g}_\lambda = \tilde{g}_{\lambda_1}\tilde{g}_{\lambda_2}\cdots\tilde{g}_{\lambda_\ell}, \qquad \hat{e}_\lambda = \hat{e}_{\lambda_1}\hat{e}_{\lambda_2}\cdots\hat{e}_{\lambda_\ell}$$

$$h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_\ell}, \qquad e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_\ell}$$

Then

$$\prod_{i,j}\frac{(tux_iy_j;q)_\infty}{(ux_iy_j;q)_\infty} = \sum_\lambda \tilde{g}_\lambda(x;q,t,u)\,m_\lambda(y)$$

$$= \sum_\lambda \frac{1}{z_\lambda(q,t,u)}\,p_\lambda(x)\,p_\lambda(y)$$

Monomial expansions

For $a=(a_{ij})\in M_{n\times\ell}(\mathbb{Z}_{\geq 0})$ let

$$rs(a)=(\mu_1,\dots,\mu_n) \qquad \text{where} \qquad \mu_i=\sum_{j=1}^{\ell} a_{ij}$$

$$cs(a)=(\lambda_1,\dots,\lambda_\ell) \qquad\qquad \lambda_j=\sum_{i=1}^{n} a_{ij}.$$

so that $rs(a)$ and $cs(a)$ are the sequences of rowsums and column sums of a, respectively

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Define

$$x^a = x^{rs(a)} = \prod_{i=1}^{n} \prod_{j=1}^{\ell} x_i^{a_{ij}} .$$

$$y^a = y^{cs(a)} = \prod_{j=1}^{\ell} \prod_{i=1}^{n} y_j^{a_{ij}} .$$

$$wt_{q,u,t}(a) = \prod_{j=1}^{\ell} \prod_{i=1}^{n} u^{a_{ij}} \frac{(tu^{-1};q)_{a_{ij}}}{(q;q)_{a_{ij}}} ,$$

where, by definition $(z;q)_0 = 1$. For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ and $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{Z}_{\geq 0}^\ell$ let

$$A_{\mu\lambda} = \{ a \in M_{n\times\ell}(\mathbb{Z}_{\geq 0}) \mid cs(a) = \lambda,\ rs(a) = \mu \}.$$

Then

$$\hat{q}_\lambda = \sum_{\mu} a_{\mu\lambda}(q,t)\, m_\mu$$

where

$$a_{\mu\lambda}(q,t) = \sum_{a \in A_{\mu\lambda}} wt_{q,t,u}(a)$$