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Binomial theorems

Using

$$\prod_{i=1}^{n} (u+x_i z) = u^n \prod_{i=1}^{n} \left(1+x_i \frac{z}{u}\right) = \sum_{r=0}^{n} u^{n-r} z^r e_r(x)$$

and specializing $x_1 = x_2 = \cdots = x_n = 1$ gives <u>the binomial theorem</u>

$$(u+z)^n = \sum_{r=0}^{n} u^{n-r} z^r \binom{n}{r} \qquad \text{and}$$

$$(u+z)^{-n} = \sum_{r\in\mathbb{Z}} u^{n-r} z^r \binom{n+r-1}{r}.$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = e_r(1,1,\ldots,1) \qquad \text{and}$$

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!} = h_r(1,1,\ldots,1).$$

Letting $x_i = q^{i-1}$ gives <u>the q-binomial theorem</u>

$$\prod_{i=1}^{n} (u+q^{i-1}z) = \sum_{r=0}^{n} q^{\frac{1}{2}r(r-1)} \begin{bmatrix} n \\ r \end{bmatrix} u^{n-r} z^r \qquad \text{and}$$

$$\prod_{i=1}^{n} \frac{1}{(u-q^{i-1}z)} = \sum_{r\in\mathbb{Z}_{\geq 0}} \begin{bmatrix} n+r-1 \\ r \end{bmatrix} u^{n-r} z^r.$$

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where

$$e_r(1,q,q^2,\dots,q^{n-1}) = \frac{(q;q)_n}{(q;q)_r(q;q)_{n-r}} = \begin{bmatrix} n \\ r \end{bmatrix} \quad \text{and}$$

$$h_r(1,q,q^2,\dots,q^{n-1}) = \frac{(q;q)_{n+r-1}}{(q;q)_r(q;q)_{n-1}} = \begin{bmatrix} n+r-1 \\ r \end{bmatrix}$$

A general infinite q-binomial theorem is

$$\frac{(tz;q)_\infty}{(uz;q)_\infty} = \prod_{i=1}^{\infty} \frac{(1-tq^{i-1}z)}{(1-uq^{i-1}z)}$$

$$= \sum_{r\in\mathbb{Z}_{\geq 0}} \left(\prod_{i=1}^{r} \frac{u-tq^{i-1}}{1-q^i} \right) z^r$$

$$= \sum_{r\in\mathbb{Z}_{\geq 0}} u^r \frac{(tu^{-1};q)_r}{(q;q)_r} z^r$$

A one sentence proof: Recognize that

$$L(z;q,t,u) = \frac{(tz;q)_\infty}{(uz;q)_\infty} \quad \text{satisfies}$$

$$L(z;q,t,u) = \frac{(1-tz)}{(1-uz)} L(qz;q,t,u)$$

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which provides a recursion on the coefficients of

$$L(z;q,t,u) = \sum_{r\in\mathbb{Z}_{\geq 0}} c_r(q,t,u)\, z^r \quad \text{as}$$

$$c_r(q,t,u)q^r - tc_{r-1}(q,t,u)q^{r-1} = c_r(q,t,u) - uc_{r-1}(q,t,u).$$

so that

$$c_r(q,t,u) = c_{r-1}(q,t,u)\,\frac{u-tq^{r-1}}{1-qq^{r-1}} = u^r\frac{(tu^{-1};q)_r}{(q;q)_r}.$$

Specializing $t=0$ and $u=0$ in the infinite q-binomial theorem give

$$\prod_{i=1}^{\infty}(1+q^{i-1}z) = (-z,q)_\infty = \sum_{r\in\mathbb{Z}_{\geq 0}} \frac{1\cdot q\cdot q^2\cdots q^{r-1}}{(q;q)_r} z^r$$

and

$$\prod_{i=1}^{\infty}\frac{1}{(1-q^{i-1}z)} = \frac{1}{(z,q)_\infty} = \sum_{r\in\mathbb{Z}_{\geq 0}}\frac{1}{(q;q)_r}z^r.$$

The finite q-binomial theorem

Put $t=q^n$ and $u=1$ in the infinite q-binomial theorem to get left hand side

$$\frac{(q^nz;q)_\infty}{(z;q)_\infty} = \frac{1}{(z;q)_n} = \prod_{i=1}^{n}\frac{1}{1-q^{i-1}z}$$

and right hand side

$$\sum_{r \in \mathbb{Z}_{\geq 0}} \frac{(q^n;q)_r}{(q;q)_r} z^r$$

with

$$(q^n;q)_r \frac{1}{(q;q)_r} = \frac{(q;q)_{n+r-1}}{(q;q)_{n-1}} \cdot \frac{1}{(q;q)_r} = \begin{bmatrix} n+r-1 \\ r \end{bmatrix}$$

so that

$$\prod_{i=1}^{n} \frac{1}{1-q^{i-1}z} = \sum_{r \in \mathbb{Z}_{\geq 0}} \frac{(q;q)_{n+r-1}}{(q;q)_r (q;q)_{n-1}} z^r$$

$$= \sum_{r \in \mathbb{Z}_{\geq 0}} \begin{bmatrix} n+r-1 \\ r \end{bmatrix} z^r.$$