The group algebra of the symmetric group S_n is the algebra generated by

$s_1, \ldots, s_{n-1}$ and $M_1, \ldots, M_n$

with relations

$M_i M_j = M_j M_i$, $M_1 = 0$, $M_i = s_{i-1} M_{i-1} s_{i-1} + s_{i-1}$

$s_i^2 = 1$, $s_i s_j = s_j s_i$ if $j \notin \{i-1, i+1\}$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

Let $q \in \mathbb{C}$.
The finite Hecke algebra $H_n^{fin}(q)$ is the algebra generated by

$T_1, \ldots, T_{n-1}$ and $Y_1, \ldots, Y_n$

with relations

$Y_i Y_j = Y_j Y_i$, $Y_1 = 1$, $Y_i = T_{i-1}^{-1} Y_{i-1} T_{i-1}^{-1}$

$T_i^2 = (q - q^{-1}) T_i + 1$, $T_i T_j = T_j T_i$ if $j \notin \{i-1, i+1\}$ and

$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$

The affine Hecke algebra $H_n^{aff}(q)$ is the algebra generated by

$T_1, \ldots, T_{n-1}$ and $Y_1, \ldots, Y_n$

with relations

$Y_i Y_j = Y_j Y_i$, $Y_i = T_{i-1}^{-1} Y_{i-1} T_{i-1}^{-1}$, $T_i^2 = (q - q^{-1}) T_i + 1$

and $T_i T_j = T_j T_i$ if $j \notin \{i-1, i+1\}$ and $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$.

Simple S_n-modules and simple $H_n^{fin}(q)$-modules

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A **simple S_n-module** is an S_n-module M with no submodules except 0 and M.

A **simple $H_n^{fin}(q)$-module** is an $H_n^{fin}(q)$ module M with no submodules except 0 and M.

Theorem (a) the map

$$\left\{\begin{matrix}\text{partitions}\\ \text{with } n \text{ boxes}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{isomorphism classes}\\ \text{of simple } S_n\text{-modules}\end{matrix}\right\}$$

$$\lambda \longmapsto S_n^\lambda$$

where S_n^λ has basis $\left\{ v_T \;\middle|\; \begin{matrix}T \text{ is a standard tableau}\\ \text{with shape } \lambda\end{matrix}\right\}$

and $\mathbb{C}S_n$-action given by

$$M_i v_T = ct(T^{-1}(i))\, v_T \quad \text{and}$$

$$\left(s_i - \frac{1}{M_i - M_{i+1}}\right) v_T = \left(1 - \frac{1}{M_i - M_{i+1}}\right) v_{s_i T}$$

(b) the map

$$\left\{\begin{matrix}\text{partitions}\\ \text{with } n \text{ boxes}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{isom. classes}\\ \text{of simple } H_n^{fin}(q)\text{-modules}\end{matrix}\right\}$$

$$\lambda \longmapsto H_n^\lambda$$

where H_n^λ has basis $\left\{ v_T \;\middle|\; \begin{matrix}T \text{ is a standard tableau}\\ \text{with shape } \lambda\end{matrix}\right\}$

and $H_n^{fin}(q)$-action given by

$$Y_i v_T = q^{ct(T^{-1}(i))}\, v_T \quad \text{and}$$

$$\left(T_i - \frac{q - q^{-1}}{1 - Y_i Y_{i+1}^{-1}}\right) v_T = \left(q^{-1} + \frac{q - q^{-1}}{1 - Y_i Y_{i+1}^{-1}}\right) v_{s_i T}.$$

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The character table of S_n and $H_k(q)$

For $\mu=(\mu_1,\ldots,\mu_\ell)$ with $\mu_1,\ldots,\mu_\ell\in\mathbb{Z}_{>0}$ define $\gamma_\mu=\gamma_{\mu_1}\times\cdots\times\gamma_{\mu_\ell}$

Theorem

$$\chi^\lambda_{q,1}(\mu)=Tr(T_{\gamma_\mu}, H_n^\lambda) \quad\text{and}$$

$$\chi^\lambda_{1,1}(\mu)=Tr(\gamma_\mu, S_n^\lambda).$$

where

$$\hat{q}_\mu=\sum_\lambda \chi^\lambda_{q,t}(\mu)\, s_\lambda \quad\text{and}$$

$$p_\mu=\sum_\lambda \chi^\lambda_{1,1}(\mu)\, s_\lambda$$

The action on $V^{\otimes k}$

Let $V=\mathbb{C}^n$ with basis $\{e_1,\ldots,e_n\}$. Then

$$V^{\otimes k}=\mathrm{span}\{v_1\otimes\cdots\otimes v_k \mid v_1,\ldots,v_k\in V\}$$
$$=\mathrm{span}\{e_{i_1}\otimes\cdots\otimes e_{i_k} \mid i_1,\ldots,i_k\in\{1,\ldots,n\}\}$$

has S_k action given by

$$s_j(e_{i_1}\cdots e_{i_k})=e_{i_1}\cdots e_{i_{j-1}}(e_{i_{j+1}}e_{i_j})e_{i_{j+2}}\cdots e_{i_k}.$$

and $H_k(q)$ action given by

$$T_j(e_{i_1}\cdots e_{i_k})=\begin{cases} q^{\frac12}e_{i_1}\cdots e_{i_{j-1}}e_{i_{j+1}}e_{i_j}e_{i_{j+2}}\cdots e_{i_k} \\ \quad +(q-1)e_{i_1}\cdots e_{i_{j-1}}e_{i_j}e_{i_{j+1}}\cdots e_{i_k} & \text{if } i_j<i_{j+1}, \\ q^{\frac12}e_{i_1}\cdots e_{i_{j-1}}e_{i_{j+1}}e_{i_j}e_{i_{j+2}}\cdots e_{i_k}. & \text{if } i_j>i_{j+1}, \\ q\,e_{i_1}\cdots e_{i_{j-1}}e_{i_j}e_{i_{j+1}}e_{i_{j+2}}\cdots e_{i_k} & \text{if } i_j=i_{j+1}. \end{cases}$$

Let $x_1, \dots, x_n \in \mathbb{C}$.

Theorem Define $D : V^{\otimes k} \to V^{\otimes k}$ by

$$D(e_{i_1} \cdots e_{i_k}) = x_{i_1} \cdots x_{i_k} e_{i_1} \cdots e_{i_k}.$$

Then

$$\mathrm{tr}(D \gamma_\mu ; V^{\otimes k}) = p_\mu(x_1, \dots, x_n)$$

$$\mathrm{Tr}(D T_{\gamma_\mu} ; V^{\otimes k}) = q_\mu(x_1, \dots, x_n ; q, 1)$$

where

$$\prod_{i=1}^{n} \frac{1 - t x_i z}{1 - u x_i z} = 1 + (u - t) \sum_{r \in \mathbb{Z}_{>0}} q_r(x_1, \dots, x_n ; u, t) z^r$$