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Adv. Disc. Math (1)
A. Ram

<u>Hecke algebras and flag varieties</u>

$G(\mathbb{F}_q^n) = \{\text{subspaces of } \mathbb{F}_q^n\}$ partially ordered by inclusion. Then

$$G(\mathbb{F}_q^n) = \bigsqcup_{k=0}^{n} G(\mathbb{F}_q^n)_k, \quad \text{where}$$

$$G(\mathbb{F}_q^n)_k = \{k\text{dimensional } \mathbb{F}_q\text{-subspaces of } \mathbb{F}_q^n\}.$$

$G(\mathbb{F}_q^n)_k$ is the <u>Grassmannian of k-~~planes~~ subspaces</u> in $\mathbb{F}_q^n$.

$G(\mathbb{F}_q^n)_1$ is the <u>projective space</u> $\mathbb{P}^{n-1}$.

The <u>flag variety</u> is

$$\begin{aligned} Fl(\mathbb{F}_q^n) &= \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{F}^n \mid \dim(V_k) = k\} \\ &= \{\text{maximal chains in } \mathbb{F}^n\} \\ &= \{\text{flags in } \mathbb{F}^n\}. \end{aligned}$$

The map

$$\begin{array}{ccc} G/B & \longrightarrow & Fl(\mathbb{F}_q^n) \\ gB & \longmapsto & gE \end{array} \quad \text{is a bijection.}$$

The map

$$\begin{array}{ccc} G/P_k & \longrightarrow & G(\mathbb{F}_q^n)_k \\ gP_k & \longmapsto & gE_k \end{array} \quad \text{is a bijection.}$$

The map

$$\begin{array}{ccc} G/P_1 & \longrightarrow & G(\mathbb{F}_q^n)_1 = \mathbb{P}^{n-1} \\ gP_1 & \longmapsto & gE_1 \end{array} \quad \text{is a bijection.}$$

If $g = \begin{pmatrix} | & & | \\ g_1 & \cdots & g_n \\ | & & | \end{pmatrix} \in GL_n(\mathbb{F}_q)$ then

$$gE_k = \mathrm{span}\left\{ \begin{pmatrix} | \\ g_1 \\ | \end{pmatrix}, \ldots, \begin{pmatrix} | \\ g_k \\ | \end{pmatrix} \right\} = \begin{bmatrix} | & & | \\ g_1 & \cdots & g_k \\ | & & | \end{bmatrix}$$

If $g = \begin{pmatrix} | & & | \\ g_1 & \cdots & g_n \\ | & & | \end{pmatrix} \in GL_n(\mathbb{F}_q)$ then

$$gE_1 = \mathrm{span}\left\{ \begin{pmatrix} | \\ g_1 \\ | \end{pmatrix} \right\} = \begin{bmatrix} | \\ g_1 \\ | \end{bmatrix}.$$

If $g = \begin{pmatrix} | & & | \\ g_1 & \cdots & g_n \\ | & & | \end{pmatrix} \in GL_n(\mathbb{F}_q)$ then

$$gE = \left(0 \subsetneq \begin{bmatrix} | \\ g_1 \\ | \end{bmatrix} \subseteq \begin{bmatrix} | & | \\ g_1 & g_2 \\ | & | \end{bmatrix} \subseteq \cdots \subseteq \begin{bmatrix} | & & | \\ g_1 & \cdots & g_n \\ | & & | \end{bmatrix} = \mathbb{F}_q^n \right)$$

Let

$$y_i(c) = 1 + (c-1)E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i}$$

and $s_i = y_i(0)$.

Then

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \text{ if } j \notin \{i-1, i+1\}, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

and

$$y_i(c_1) y_i(c_2) = \begin{cases} y_i(c_1 + c_2^{-1}) h_i(c_2) h_{i+1}(-c_2^{-1}) x_{i,i+1}(c_2^{-1}), & \text{if } c_2 \neq 0, \\ x_{i,i+1}(c_1), & \text{if } c_2 = 0 \end{cases}$$

$$y_i(c_1) y_j(c_2) = y_j(c_2) y_i(c_1), \quad \text{if } j \notin \{i-1, i+1\},$$

$$y_i(c_1) y_{i+1}(c_2) y_i(c_3) = y_{i+1}(c_3) y_i(c_1 c_3 + c_2) y_{i+1}(c_1).$$

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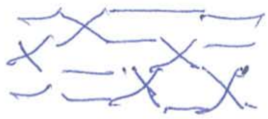
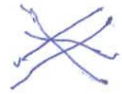
Example $s_2 s_1 s_3 s_2 s_3 =$  $=$ 

$$\begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix} = s_2 \begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0 \end{pmatrix} = s_2 s_1 \begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0 \end{pmatrix}$$

$$= s_2 s_1 s_3 \begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&1&0&0\\ 0&0&1&0 \end{pmatrix} = s_2 s_1 s_3 s_2 \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix} = s_2 s_1 s_3 s_2 s_3 \begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix}$$

Example

$$\begin{pmatrix} 7&6&2&4\\ 1&8&7&9\\ 8&6&3&5\\ 0&1&1&2 \end{pmatrix} = y_2\left(\tfrac{1}{8}\right) \begin{pmatrix} 7&6&2&4\\ 8&6&3&5\\ 0&\frac{58}{8}&\frac{53}{8}&\frac{67}{8}\\ 0&1&1&2 \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) \begin{pmatrix} 8&6&3&5\\ 0&\frac{3}{4}&\frac{1}{4}&-\frac{3}{8}\\ 0&\frac{58}{8}&\frac{53}{8}&\frac{67}{8}\\ 0&1&1&2 \end{pmatrix} = y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{29}{4}\right) \begin{pmatrix} 8&6&3&5\\ 0&\frac{3}{4}&\frac{1}{4}&-\frac{3}{8}\\ 0&1&1&2\\ 0&0&-\frac{5}{8}&-\frac{49}{8} \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{29}{4}\right) y_2\left(\tfrac{3}{4}\right) \begin{pmatrix} 8&6&3&5\\ 0&1&1&2\\ 0&0&-\frac{1}{2}&-\frac{15}{8}\\ 0&0&-\frac{5}{8}&-\frac{49}{8} \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{29}{4}\right) y_2\left(\tfrac{3}{4}\right) y_3\left(\tfrac{4}{5}\right) \begin{pmatrix} 8&6&3&5\\ 0&1&1&2\\ 0&0&-\frac{5}{8}&-\frac{49}{8}\\ 0&0&0&-\frac{71}{40} \end{pmatrix}$$

Now

$$y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{29}{4}\right) y_2\left(\tfrac{3}{4}\right) y_3\left(\tfrac{4}{5}\right)$$

$$= x_{23}\left(\tfrac{1}{8}\right) s_2\, x_{12}\left(\tfrac{7}{8}\right) s_1\, x_{34}\left(\tfrac{29}{4}\right) s_3\, x_{23}\left(\tfrac{3}{4}\right) s_2\, x_{34}\left(\tfrac{4}{5}\right) s_3$$

$$= x_{23}\left(\tfrac{1}{8}\right) x_{13}\left(\tfrac{7}{8}\right) s_2 s_1\, x_{34}\left(\tfrac{29}{4}\right) s_3\, x_{23}\left(\tfrac{3}{4}\right) s_2\, x_{34}\left(\tfrac{4}{5}\right) s_3$$

$$= x_{23}\left(\tfrac{1}{8}\right) x_{13}\left(\tfrac{7}{8}\right) x_{24}\left(\tfrac{29}{4}\right) s_2 s_1 s_3\, x_{23}\left(\tfrac{3}{4}\right) s_2\, x_{34}\left(\tfrac{4}{5}\right) s_3$$

$$= x_{23}\left(\tfrac{1}{8}\right) x_{13}\left(\tfrac{7}{8}\right) x_{24}\left(\tfrac{29}{4}\right) x_{14}\left(\tfrac{3}{4}\right) s_2 s_1 s_3 s_2\, x_{34}\left(\tfrac{4}{5}\right) s_3$$

$$= x_{23}\left(\tfrac{1}{8}\right) x_{13}\left(\tfrac{7}{8}\right) x_{24}\left(\tfrac{29}{4}\right) x_{14}\left(\tfrac{3}{4}\right) x_{12}\left(\tfrac{4}{5}\right) s_2 s_1 s_3 s_2 s_3$$

So

$$\begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 5 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} \in B s_2 s_1 s_3 s_2 s_3 B.$$

<u>Theorem</u> <u>4.8</u> Let B be the group of upper triangular invertible matrices. Let $g \in GL_n(\mathbb{F})$. Then there exists a unique $w \in S_n$ and unique $c_1, \dots, c_\ell \in \mathbb{F}$ and a unique $b \in B$ such that

$$g = y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell)\, b$$

where $w = s_{i_1} \cdots s_{i_\ell}$ is the greedy reduced word for w.