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Adv. Discrete Math.
A. Ram

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The binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Application: Prove that if $xy = yx$ then

$$e^{x+y} = e^x e^y$$

Differential equations

$$\frac{d\,(1+x)^n}{dx} = \frac{n}{(1+x)}(1+x)^n \quad \text{and} \quad \frac{d e^x}{dx} = e^x.$$

More generally

$$(1-z)^{-\alpha} = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(\alpha)_k}{k!} z^k, \quad \text{where}$$

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) \text{ and } (a;q)_k = (1-a)(1-aq)\cdots(1-q^{k-1}a)$$

The q-hypergeometric series ${}_{r+1}\varphi_r$ is defined by

$${}_{r+1}\varphi_r\left[\begin{matrix} a_0\, a_1 \cdots a_r \\ b_1 \cdots b_r \end{matrix}; q, z\right] = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(a_0;q)_k (a_1;q)_k \cdots (a_r;q)_k}{(q;q)_k (b_1;q)_k \cdots (b_r;q)_k} z^k.$$

The generalized hypergeometric series is ${}_{r+1}F_r$ given by

$${}_{r+1}F_r\left[\begin{matrix} \alpha_0\, \alpha_1 \cdots \alpha_r \\ \beta_1 \cdots \beta_r \end{matrix}; z\right] = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(\alpha_0)_k (\alpha_1)_k \cdots (\alpha_r)_k}{(1)_k (\beta_1)_k \cdots (\beta_r)_k} z^k.$$

q-analogues of the binomial theorem

$$(1+x)(1+xq)\cdots(1+xq^{n-1}) = \sum_{k=0}^{n} x^k q^{\binom{k}{2}} \mathrm{Card}(G(\mathbb{F}_q^n)_k)$$
$$= \sum_{k=0}^{n} x^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}$$

or
$$\prod_{i=1}^{n} (u + q^{i-1}z) = \sum_{r=0}^{n} q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix} u^{n-r} z^r$$
$$= \sum_{r=0}^{n} q^{\binom{r}{2}} e_r(1,q,q^2,\ldots,q^{n-1}) u^{n-r} z^r.$$

Then
$$\prod_{i=1}^{n} \frac{1}{(u-q^{i-1}z)} = \sum_{r\in\mathbb{Z}_{\geq 0}} \begin{bmatrix} n+r-1 \\ r \end{bmatrix} u^{n-r} z^r$$

$$= \sum_{r\in\mathbb{Z}_{\geq 0}} h_r(1,q,q^2,\ldots,q^{n-1}) u^{n-r} z^r$$

and
$$\frac{(tz;q)_\infty}{(uz;q)_\infty} = \prod_{i=1}^{\infty} \frac{(1-tq^{i-1}z)}{(1-uq^{i-1}z)} = \sum_{r\in\mathbb{Z}_{\geq 0}} \frac{(tu^{-1};q)_r}{(q;q)_r} u^r z^r.$$

Difference equation: If $L(z;q,t,u) = \dfrac{(tz;q)_\infty}{(uz;q)_\infty}$

then $$L(z;q,t,u) = \frac{(1-tz)}{(1-uz)} L(qz;q,t,u)$$

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Power functions and exponential functions

Let
$$[a] = \frac{1-q^a}{1-q} \quad\text{and}\quad [n]! = [n][n-1]\cdots[2][1]$$

The 'power function': If $x = 1-z$ then
$$x^{-\alpha} = (1-z)^{-\alpha} = \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} z^n$$
$$= \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{(\alpha)_n}{n!} z^n = {}_1F_0(\alpha; z)$$

Differential equation:
$$\frac{df}{dx} = \frac{-\alpha}{x} f \quad\text{or}\quad (1-z)\frac{df}{dx} = af$$

The 'q-power function':
$$[(1-z)^{-a}] = \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{[a][a+1]\cdots[a+n-1]}{[n]!}$$
$$= \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{(a;q)_n}{[n]!} z^n = {}_1\varphi_0(a; q, z).$$

Difference equation:
$$(1-zq^a)\,g(qz) = (1-z)\,g(z)$$

Product formula:
$$[(1-z)^{-a}] = \prod_{n\in\mathbb{Z}_{\geq 0}} \frac{1-zq^{n+a}}{1-zq^n} = \frac{(zq^a;q)_\infty}{(z;q)_\infty}$$

The exponential function

$$\exp(z) = \sum_{n\in\mathbb{Z}_{\ge 0}} \frac{1}{n!} z^n \quad \text{and} \quad \exp(-z) = \sum_{n\in\mathbb{Z}_{\ge 0}} \frac{(-1)^n}{n!} z^n$$

Differential equation:

$$\frac{df}{dz} = f$$

The q-exponential functions

$$(z;q)_\infty^{-1} = \sum_{n\in\mathbb{Z}_{\ge 0}} \frac{1}{(q;q)_n} z^n, \quad (z;q)_\infty = \sum_{n\in\mathbb{Z}_{\ge 0}} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} z^n.$$

Product formulas:

$$(z;q)_\infty^{-1} = \prod_{k\in\mathbb{Z}_{\ge 0}} \frac{1}{1-zq^k} \qquad (z;q)_\infty = \prod_{k\in\mathbb{Z}_{\ge 0}} (1-zq^k)$$

Difference equations:

$$(qz;q)_\infty^{-1} = (1-z)(z;q)_\infty^{-1} \qquad (qz;q)_\infty = \frac{1}{1-z}(z;q)_\infty.$$