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The gamma function

Integral formula

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt$$

Product formula

$$\Gamma(a) = \prod_{n=1}^{\infty} \frac{n}{n+a} \left(\frac{n+1}{n} \right)^a$$

Difference equation

$$\Gamma(a+1) = a\Gamma(a)$$

(so $\Gamma(r) = (r-1)!$ if $r \in \mathbb{Z}_{\geq 0}$)

The q-gamma function

Integral formula

$$\Gamma_q(a) = \int_0^{\frac{1}{1-q}} t^{a-1} E_q(-qt)\, d_q t$$

Product formula

$$\Gamma_q(a) = \prod_{n=1}^{\infty} \frac{[n]}{[n+a]} \left(\frac{[n+1]}{[n]} \right)^a = (1-q)^{1-a} \frac{(q;q)_\infty}{(q^a;q)_\infty}$$

or $\Gamma_q(r) = \dfrac{(q;q)_{r-1}}{(1-q)^{r-1}}$

Difference equation

$$\Gamma_q(a+1) = [a]\, \Gamma_q(a)$$

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The beta function

Integral formula

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^y \frac{dt}{(1-t)}$$

Product formula $\beta(x,y)$ is an analogue of a binomial coefficient since

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Difference equation

$$\beta(x,y) = \beta(x,y+1) + \beta(x+1,y),$$

$$\beta(x+1,y) = \beta(x,y)\frac{x}{x+y} \quad \text{and} \quad \beta(x,y+1) = \beta(x,y)\frac{y}{x+y}$$

The q-beta function

Integral formula

$$B_q(r,s) = \int_0^1 t^{r-1} \frac{1}{[(1-t)^{-s}]} \frac{d_qt}{(1-t)}$$

$$= \int_0^1 t^{r-1}(qx;q)_{s-1}\, d_qx$$

Difference equation

$$q^y B_q(x+1,y) + B_q(x,y+1) = B_q(x,y),$$

$$B_q(x+1,y) = \frac{1-q^x}{1-q^{x+y}} B_q(x,y) \quad \text{and} \quad B_q(x,y+1) = \frac{1-q^y}{1-q^{x+y}} B_q(x,y).$$

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The Gauss hypergeometric function

Power series:

$${}_2F_1(a,b,c;z) = \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$.

Integral formula

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt$$

Differential equation

$$z(1-z)\frac{d^2F}{dz^2} + (c-(a+b-1)z)\frac{dF}{dz} - abF = 0$$

Then ${}_2F_1(a,b,c;z)$ is a solution with $F(0)=1$.
General solutions are

$$F = c_1\, {}_2F_1(a,b,c;z) + c_2\, {}_2F_1(a-c+1, b-c+1, 2-c; z)$$

for $c_1, c_2 \in \mathbb{C}$.

The Picard-Fuchs equations for a family of elliptic curves are

$$\lambda(\lambda-1)\frac{d^2f}{d\lambda^2} + (2\lambda-1)\frac{df}{d\lambda} + \frac{1}{4}f = 0$$

and the conversion is

$$ab = \frac{1}{4}, \quad \frac{c}{a+b+1} = \frac{1}{2}, \quad \lambda = -z \text{ and } f = F.$$

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The q-hypergeometric function

Power series:

$$ {}_2\phi_1(q^a, q^b; q^c; z) = \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{(q^a; q)_n (q^b; q)_n}{(q^c; q)_n (q; q)_n} z^n $$

Integral formula

$$ {}_2\phi_1(a, b, c; z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{[(1-tz)^{-a}]}{[(1-tz)^{b-a}]} \frac{d_q t}{(1-t)} $$

Difference equation

$$ (q^{a+b} z - q^{c-1}) \phi(q^2 z) + (-(q^a + q^b) z + q^{c-1} + 1) + (z-1) \phi(z) = 0. $$