1.24 The fundamental theorem of calculus: interpreting the limit via areas

If

$$\frac{df}{dx} = g$$

then define

$$\frac{df}{dx}\Big]_{x=a} = g(a) \qquad \text{and} \qquad \left(\int g\,dx\right)\Big]_{x=a}^{x=b} = f(b) - f(a).$$

Fundamental theorem of change.

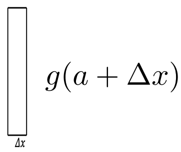
$$\frac{df}{dx}\Big]_{x=a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Fundamental theorem of calculus.

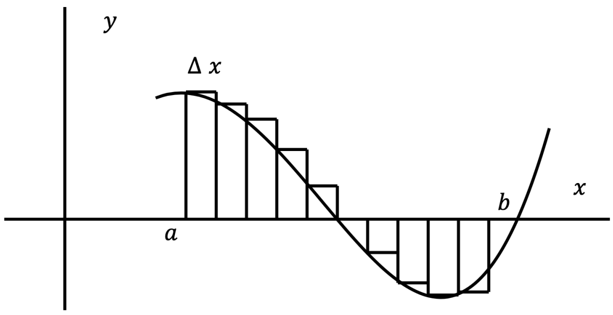
$$\left(\int g\,dx\right)\Big]_{x=a}^{x=b} = \lim_{N\to\infty} \left(g(a)\tfrac{1}{N} + g(a+\tfrac{1}{N})\tfrac{1}{N} + \cdots + g(b-\tfrac{1}{N})\tfrac{1}{N}\right).$$

The right hand side

$$\begin{aligned} &\lim_{N\to\infty} \left(g(a)\tfrac{1}{N} + g(a+\tfrac{1}{N})\tfrac{1}{N} + \cdots + g(b-\tfrac{1}{N})\tfrac{1}{N}\right) \\ &\qquad = \lim_{N\to\infty} (\text{add up the areas of the little boxes of width } \Delta x = \tfrac{1}{N} \text{ and height } g(a+k\tfrac{1}{N})) \end{aligned}$$



little box



How little boxes are used to calculate an integral

The leftmost box has area $g(a)\Delta x = g(a)\frac{1}{N}$.
The second box has area $g(a+\Delta x)\Delta x = g(a+\frac{1}{N})\frac{1}{N}$.
Continue this process.
So think of $\lim_{N\to\infty} \left(g(a)\frac{1}{N} + g(a+\frac{1}{N})\frac{1}{N} + \cdots + g(b-\frac{1}{N})\frac{1}{N}\right)$ as adding up areas from a to b of infinitesimally small boxes with area $g(x)\Delta x$.