2 Some lectures

2.1 Calculus, Functions and inverse functions

Calculus is the study of

(1) Derivatives	(3) Applications of derivatives
(2) Integrals	(4) Applications of integrals

A *derivative* is a creature you put a function into, it chews on it, and spits out a new function.

$$f \to \boxed{\frac{d}{dx}} \to \frac{df}{dx}.$$

The *integral* is the derivative backwards:

$$f \leftarrow \boxed{\int \quad dx} \leftarrow \frac{df}{dx} \qquad \text{or} \qquad \frac{df}{dx} \to \boxed{\int \quad dx} \to f.$$

A *function* is one down on the food chain.

$$\begin{matrix} \text{input} & & & & \text{output} \\ \text{number} & \to & \boxed{f} & \to & \text{number} \\ x & & & & f(x) \end{matrix}$$

Functions take a number as input, chew on it a bit, and spit out a new number.

The *inverse function* to f is f backwards:

$$x \leftarrow \boxed{f^{-1}} \leftarrow f(x) \qquad \text{or} \qquad \begin{matrix} f(x) \to & & \to x \\ & \boxed{f^{-1}} & \\ z \to & & \to f^{-1}(z) \end{matrix}$$

Example.

	$f(x) = x^2$			The inverse function is $f^{-1}(x) = \sqrt{x}$	
$x \to$		$\to x^2$	$x^2 \to$		$\to x$
$1 \to$		$\to 1$	$1 \to$		$\to 1$
$2 \to$		$\to 4$	$4 \to$		$\to 2$
$3 \to$		$\to 9$	$9 \to$		$\to 3$
$-3 \to$		$\to 9$	$9 \to$		$\to -3$
$\pi \to$		$\to \pi^2$	$\pi^2 \to$		$\to \pi$
$\sqrt{7} \to$		$\to 7$	$7 \to$		$\to \sqrt{7}$

The inverse function is not always a function because there might be some uncertainty about what the inverse function will spit out:

$$9 \to \boxed{f^{-1}(x) = \sqrt{x}} \to 3 \qquad \text{or} \qquad 9 \to \boxed{f^{-1}(x) = \sqrt{x}} \to -3.$$

Numbers are at the very bottom of the food chain.

2.1.1 And so we discovered ... Numbers

At some point humankind wanted to count things and discovered the **positive integers**,

$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots.$$

GREAT for counting something,
BUT what if you don't have anything? How do we talk about nothing, nulla, zilch?
... and so we discovered the **nonnegative integers**,

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots.$$

GREAT for adding,

$$5+3=8,\ \ 0+10=10,\ \ 21+37=48,$$

BUT not so great for subtraction,

$$5-3=2,\ \ 2-0=2,\ \ 12-34=???.$$

... and so we discovered the **integers**

$$\ldots,-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\ .$$

GREAT for adding, subtracting and multiplying,

$$3\cdot 6=18,\ \ -3\cdot 2=-6,\ \ 0\cdot 7=0,$$

BUT not so great if you only want part of the sausage ...,
... and so we discovered the **rational numbers**,

$$\frac{a}{b}\ , \qquad a \text{ an integer, } b \text{ an integer, } b\neq 0.$$

GREAT for addition, subtraction, multiplication, and division,
BUT not so great for finding $\sqrt{2}=????$,
... and so we discovered the **real numbers**,

all decimal expansions.

Examples:

$$\begin{aligned} \pi &= 3.1415926\ldots\ , \\ e &= 2.71828\ldots\ , \\ \sqrt{2} &= 1.414\ldots\ , \\ 10 &= 10.0000\ldots\ , \end{aligned} \qquad \begin{aligned} \tfrac{1}{3} &= .3333\ldots\ , \\ \tfrac{1}{8} &= .125 = .125000000\ldots\ , \end{aligned}$$

GREAT for addition, subtraction, multiplication, and division,
BUT not so great for finding $\sqrt{-9}=????$,
... and so we discovered the **complex numbers**,

$$a+bi, \qquad a \text{ a real number, } b \text{ a real number, } i=\sqrt{-1}.$$

2.1.2 Operations on complex numbers

Examples of complex numbers: $3+\sqrt{2}i$, $6 = 6+0i$, $\pi+\sqrt{7}i$,
and

$$\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9}\sqrt{-1} = 3i.$$

GREAT.

Addition: $(3+4i)+(7+9i) = 3+7+4i+9i = 10+13i$.

Subtraction: $(3+4i)-(7+9i) = 3-7+4i-9i = -4-5i$.

Multiplication:

$$\begin{aligned}(3+4i)(7+9i) &= 3(7+9i)+4i(7+9i)\\ &= 21+27i+28i+36i^2\\ &= 21+55i-36\\ &= -15+55i.\end{aligned}$$

Division:

$$\begin{aligned}\frac{3+4i}{7+9i} &= \frac{(3+4i)}{(7+9i)}\frac{(7-9i)}{(7-9i)} = \frac{21-27i+28i+36}{49-63i+63i+81}\\ &= \frac{57+i}{130} = \frac{57}{130}+\frac{1}{130}i.\end{aligned}$$

Square Roots: We want $\sqrt{-3+4i}$ to be some $a+bi$.

$$\text{If} \qquad \sqrt{-3+4i} = a+bi$$

then

$$\begin{aligned}-3+4i = (a+bi)^2 &= a^2+abi+abi+b^2i^2\\ &= a^2-b^2+2abi.\end{aligned}$$

So

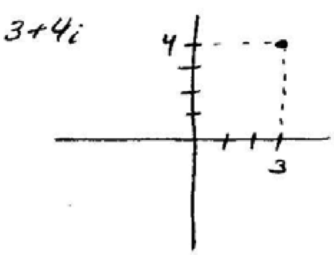
$$a^2-b^2 = -3 \qquad \text{and} \qquad 2ab = 4.$$

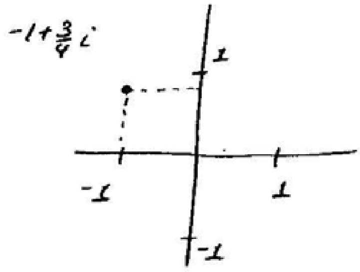
Solve for a and b.

$$\begin{aligned} b = \frac{4}{2a} = \frac{2}{a}. \qquad &\text{So} \quad a^2 - \left(\frac{2}{a}\right)^2 = -3.\\ &\text{So} \quad a^2 - \frac{4}{a^2} = -3.\\ &\text{So} \quad a^4 - 4 = -3a^2.\\ &\text{So} \quad a^4+3a^2-4 = 0.\\ &\text{So} \quad (a^2+4)(a^2-1) = 0.\end{aligned}$$

So $a^2 = -4$ or $a^2 = 1$.
So $a = \pm 1$, and $b = \frac{2}{\pm 1} = 2$ or -2.
So $a+bi = 1+2i$ or $a+bi = -1-2i$.
So $\sqrt{-3+4i} = \pm(1+2i)$.

Graphing:





Really, the i-axis and not-i-axis should be properly labeled

Factoring:

$$x^2 + 5 = (x + \sqrt{5}\,i)(x - \sqrt{5}\,i),$$
$$x^2 + x + 1 = \left(x - \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\right)$$

This is REALLY why we like the complex numbers.
The **fundamental theorem of algebra** says that ANY POLYNOMIAL

$$\text{(for example, } x^{12673} + 2563x^{159} + \pi x^{121} + \sqrt{7}\,x^{23} + 9621\tfrac{1}{2})$$

can be factored completely as

$$(x - u_1)(x - u_2)\cdots(x - u_n)$$

where $u_1, u_2, \ldots, u_n$ are complex numbers.

2.2 Limits

$$\lim_{x\to 2} f(x) = 10 \qquad \text{if } f(x) \text{ gets closer and closer to 10 as } x \text{ gets closer and closer to 2.}$$

Example. Evaluate $\lim_{x\to 2} \frac{3x^2+8}{x^2-x}$.

When $x = 1$, $\quad \frac{3x^2+8}{x^2-x} = 11.$

When $x = 1.5$, $\quad \frac{3x^2+8}{x^2-x} = 19.66...$

When $x = 1.9$, $\quad \frac{3x^2+8}{x^2-x} = 11.011...$

When $x = 1.99$, $\quad \frac{3x^2+8}{x^2-x} = 10.091...$

When $x = 1.999$, $\quad \frac{3x^2+8}{x^2-x} = 10.00901...$

When $x = 1.9999$, $\quad \frac{3x^2+8}{x^2-x} = 10.0009001...$

So

$$\lim_{x\to 2} \frac{3x^2+8}{x^2-x} = 10.$$