4 The "number systems" $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^2$, $\mathbb{R}^n$

4.1 Numbers and intervals

The positive integers: $\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$.

The nonnegative integers: $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$.

The integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

The rational numbers: $\mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{Z}_{\neq 0} \text{ and } \frac{a}{b} = \frac{c}{d} \text{ if } ad = bc \right\}$.

The real numbers:

$$\mathbb{R} = \{\pm a_\ell a_{\ell-1} \ldots a_1 a_0 . a_{-1} a_{-2} \ldots \mid \ell \in \mathbb{Z}_{\geq 0},\ a_i \in \{0, \ldots, 9\},\ a_\ell \neq 0 \text{ if } \ell > 0\}.$$

with a requirement that if $a_k \neq 9$ then $\pm a_\ell ... a_{k+1} a_k 9999... = \pm a_\ell ... a_{k+1}(a_k + 1)000...$ so that, for example, $0.9999... = 1.0000....$

The complex numbers:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\} \qquad \text{with } i^2 = -1.$$

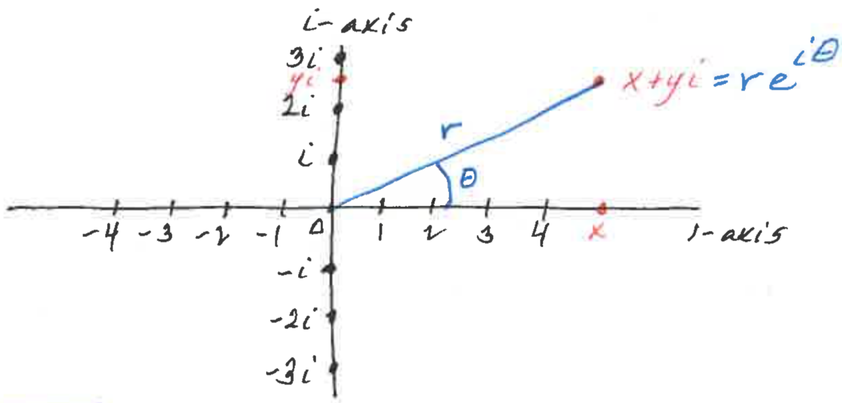
Let $a, b \in \mathbb{R}$ with $a < b$. Define

$$\mathbb{R}_{(a,b)} = \{x \in \mathbb{R} \mid a < x < b\}, \qquad \mathbb{R}_{[a,b)} = \{x \in \mathbb{R} \mid a \leq x < b\}$$

$$\mathbb{R}_{(a,b]} = \{x \in \mathbb{R} \mid a < x \leq b\}, \qquad \mathbb{R}_{[a,b]} = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

$$\mathbb{R}_{(a,\infty)} = \{x \in \mathbb{R} \mid a < x\}, \qquad \mathbb{R}_{[a,\infty)} = \{x \in \mathbb{R} \mid a \leq x\}$$

$$\mathbb{R}_{(\infty,a)} = \{x \in \mathbb{R} \mid x < a\}, \qquad \mathbb{R}_{(\infty,a]} = \{x \in \mathbb{R} \mid x \leq a\}.$$



Picture of $\mathbb{Z} \subseteq \mathbb{R} \subseteq \mathbb{C}$

What does $\frac{1}{a}$ really mean?

$\frac{1}{a}$ is the number that when multiplied by a gives 1.

4.6 The complex numbers $\mathbb{C}$

The *complex numbers* is the $\mathbb{R}$-algebra

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\} \qquad \text{with} \quad i^2 = -1,$$

so that if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned} \qquad \text{and} \qquad \begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + i(x_1y_2 + x_2y_1) + i^2 y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \end{aligned}$$

The *complex conjugation*, or *Galois automorphism*, is the function

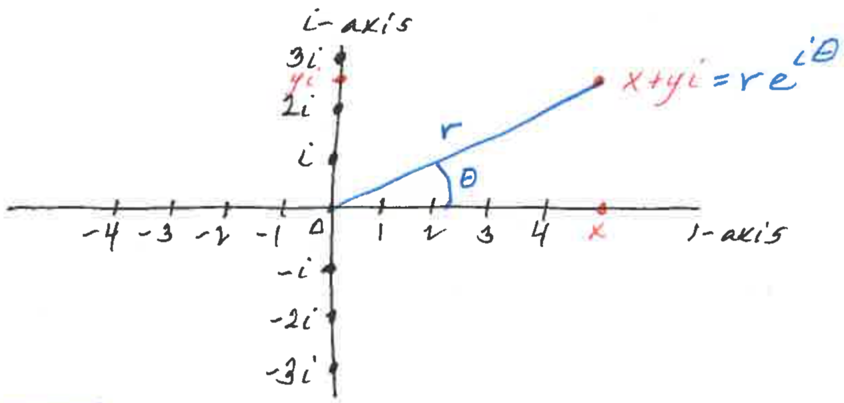
$$\overline{} \colon \mathbb{C} \to \mathbb{C} \qquad \text{given by} \quad \overline{x + iy} = x - iy.$$

The *norm*, or *length function*, on $\mathbb{C}$ is the function

$$|\ | \colon \mathbb{C} \to \mathbb{R}_{\geq 0} \qquad \text{given by} \quad |x + iy| = \sqrt{x^2 + y^2}.$$

The *Hermitian form*, or *inner product*, on $\mathbb{C}$ is

$$\langle , \rangle \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C} \qquad \text{given by} \quad \langle z_1, z_2 \rangle = z_1 \overline{z_2}.$$



Graphing complex numbers

If $r \in \mathbb{R}_{\geq 0}$ and $\theta \in \mathbb{R}$ then

$$re^{i\theta} = r\cos\theta + ir\sin\theta.$$

If $z \in \mathbb{C}$ and $z \neq 0$ then

$$z^{-1} = \frac{1}{|z|^2}\overline{z},$$

since if $z = x + iy$ then

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{1}{(x + iy)}\frac{(x - iy)}{(x - iy)} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

HW: Show that if $z \in \mathbb{C}$ then $|z|^2 = z\overline{z}$.
HW: Show that if $z_1, z_2 \in \mathbb{C}$ then $|z_1 z_2| = |z_1|\,|z_2|$.