4.8 The favourite space $\mathbb{R}^2$

The favourite example is $\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$ with *addition* and *scalar multiplication* given by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \qquad \text{and} \qquad c(x_1, x_2) = (cx_1, cx_2), \quad \text{for } c \in \mathbb{R},$$

with *inner product*

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}_{\geq 0} \\ (x, y) & \longmapsto & \langle x, y \rangle \end{array} \qquad \text{given by} \qquad \langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + x_2 y_2,$$

with *norm*

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}_{\geq 0} \\ x & \longmapsto & \|x\| \end{array} \qquad \text{given by} \qquad \|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2},$$

with *metric* $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ given by

$$d((x_1, x_2), (y_1, y_2)) = \|(x_1, x_2) - (y_1, y_2)\| = \|(x_1 - y_1, x_2 - y_2)\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

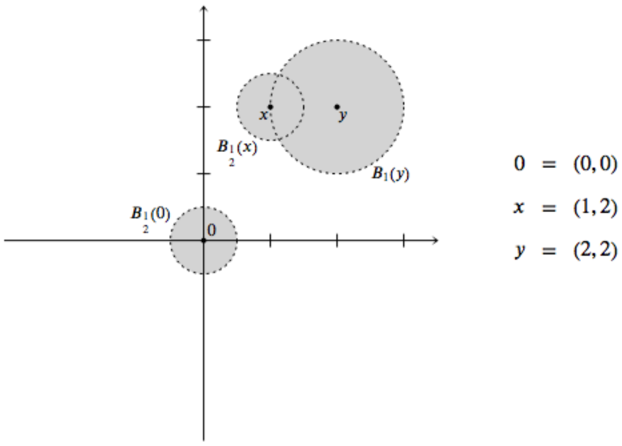
with *angle function* $\theta\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{[0,2\pi)}$ given by

$$\theta((x_1, x_2), (y_1, y_2)) = \arccos\Big(\frac{\langle (x_1, x_2), (y_1, y_2) \rangle}{\|(x_1, x_2)\| \cdot \|(y_1, y_2)\|} \Big),$$

and

$$B_\epsilon(x) = \{y \in \mathbb{R}^2 \mid d(y, x) < \epsilon\}$$

is the *ball of radius ϵ centered at x* (yes, to stress, strongly, that we normally assume that the set $\mathbb{R}^2$ is endowed with lots of extra structures this is, intentionally, a very run-on sentence).



Open balls in $\mathbb{R}^2$.

4.9 The favourite spaces $\mathbb{R}^n$

4.9.1 n-tuples are functions

Let $n \in \mathbb{Z}_{>0}$. Identify n-tuples $(x_1, \ldots, x_n)$ of elements of $\mathbb{R}$ with functions $\vec{x}\colon \{1, 2, \ldots, n\} \to \mathbb{R}$ so that

$$\text{the } n\text{-tuple } (x_1, \ldots, x_n) \qquad \text{is identified with the function} \quad \begin{array}{cccc} \vec{x}\colon & \{1, \ldots, n\} & \to & \mathbb{R} \\ & i & \mapsto & x_i \end{array}$$

4.9.2 The vector space $\mathbb{R}^n$

Let $n \in \mathbb{Z}_{\geq 0}$. The space of functions from $\{1, 2, \dots, n\}$ to $\mathbb{R}$ is

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}\} = \{\text{functions } \vec{x}\colon \{1, \dots, n\} \to \mathbb{R}\}.$$

with *addition and scalar multiplication* given by

$$\begin{aligned} (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \quad \text{and} \\ c(x_1, x_2, \dots, x_n) &= (cx_1, cx_2, \dots, x_n), \qquad \text{for } c \in \mathbb{R}, \end{aligned}$$

and with *inner product* $\langle\, , \rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle = x_1 y_1 + x_2 y_2 + \cdots x_n y_n,$$

with *norm*

$$\begin{matrix} \mathbb{R}^n & \longrightarrow & \mathbb{R}_{\geq 0} \\ x & \longmapsto & \|x\| \end{matrix} \qquad \text{given by} \qquad \|(x_1, x_2, \dots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

with *metric* $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$\begin{aligned} &d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \|(x_1, x_2, \dots, x_n) - (y_1, y_2, \dots, y_n)\| \\ &\quad = \|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}, \end{aligned}$$

with *angle function* $\theta\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{[0,2\pi)}$ given by

$$\theta((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \arccos\Big(\frac{\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle}{\|(x_1, x_2, \dots, x_n)\| \cdot \|(y_1, y_2, \dots, y_n)\|} \Big),$$

and

$$B_\epsilon(x) = \{y \in \mathbb{R}^n \mid d(y, x) < \epsilon\}$$

is the *ball of radius* ϵ *centered at* x (yes, to stress, strongly, that we normally assume that the set $\mathbb{R}^n$ is endowed with lots of extra structures this is, intentionally, a very run-on sentence).