26.03.2015 ①
Calculus Lect. 10
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2.49 and 2.50 Euler's formula

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$ | $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$e^{-i\theta} = \cos(\theta) - i\sin(\theta)$ | $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$

So $e^{i\theta} + e^{-i\theta} = 2\cos(\theta)$ | $e^{i\theta} - e^{-i\theta} = 2i\sin(\theta)$

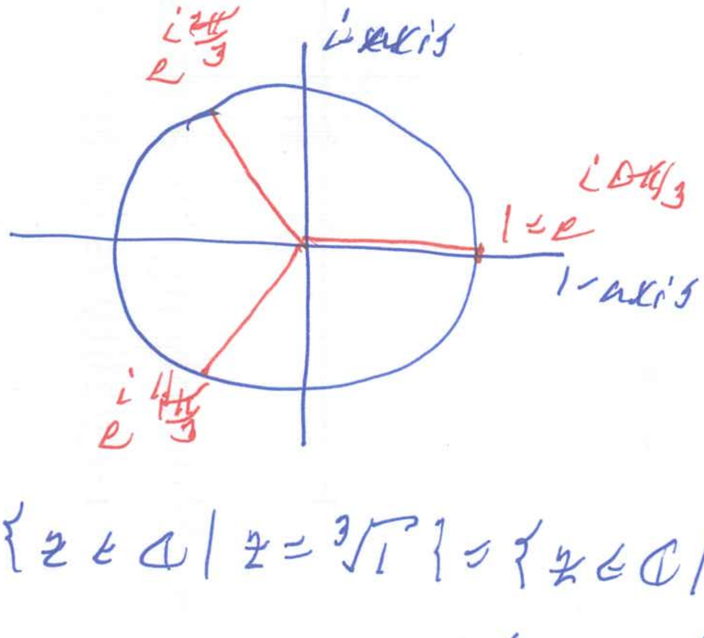
So $\frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \cos(\theta)$ | $\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \sin(\theta)$.

Cube roots of 1

$$\{z \in \mathbb{C} \mid z = \sqrt[3]{1}\} = \{z \in \mathbb{C} \mid z = 1^{1/3}\}$$

$$= \{z \in \mathbb{C} \mid z = (e^{i0\pi})^{1/3} \text{ or } z = (e^{i2\pi})^{1/3} \text{ or } z = (e^{i(4\pi)})^{1/3}, \dots\}$$

$$= \{e^{\frac{i0\pi}{3}}, e^{i2\pi/3}, e^{\frac{i4\pi}{3}}, \dots\} = \{1, e^{i2\pi/3}, e^{i4\pi/3}\}$$



$$\{z \in \mathbb{C} \mid z = \sqrt[3]{1}\} = \{z \in \mathbb{C} \mid z = 1^{1/3}\}$$

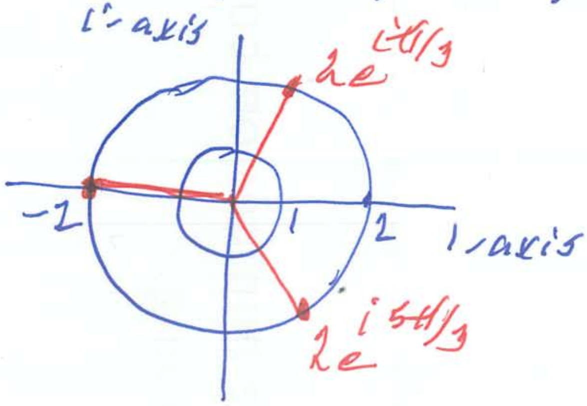
$$= \{z \in \mathbb{C} \mid z^3 = 1\}$$

$$= \{z \in \mathbb{C} \mid z^3 - 1 = 0\}$$

$$= \{z \in \mathbb{C} \mid (z-1)(z - e^{i2\pi/3})(z - e^{i4\pi/3}) = 0\}$$

Cube roots of −8

$$\{z\in\mathbb{C} \mid z = \sqrt[3]{-8}\} = \{z\in\mathbb{C}\mid z=(-8)^{1/3}\}$$
$$= \{z\in\mathbb{C}\mid z = (e^{i\pi}\cdot 8)^{1/3}\} = \{z\in\mathbb{C}\mid z = (2^3(e^{i\pi/3})^3\cdot 1)^{1/3}\}$$
$$= \{z\in\mathbb{C}\mid z = 2e^{i\pi/3}\cdot 1^{1/3}\}$$
$$= \{2e^{i\pi/3}\cdot 1,\ 2e^{i\pi/3}\cdot e^{i2\pi/3},\ 2e^{i\pi/3}e^{i4\pi/3}\}$$
$$= \{2e^{i\pi/3},\ 2e^{i\pi},\ 2e^{i5\pi/3}\}$$

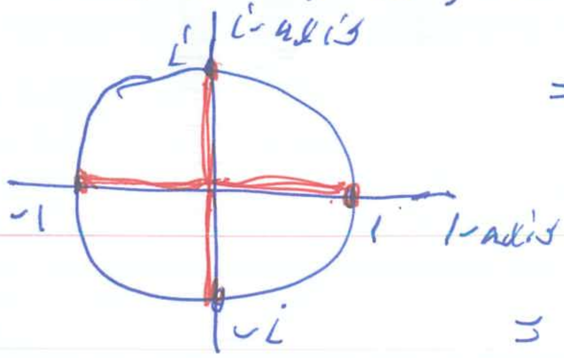


Check:

$(-2)^3 = (-2)(-2)(-2) = 8$

4th roots of 1

$$\{z\in\mathbb{C}\mid z=\sqrt[4]{1}\} = \{z\in\mathbb{C}\mid z = 1^{1/4}\}$$
$$= \{(e^{i0\pi})^{1/4}, (e^{i2\pi})^{1/4}, (e^{i4\pi})^{1/4}, (e^{i6\pi})^{1/4}, \dots\}$$
$$= \{e^0, e^{i\pi/2}, e^{i\pi}, e^{i3\pi/2}\} = \{1, i, -1, -i\}$$

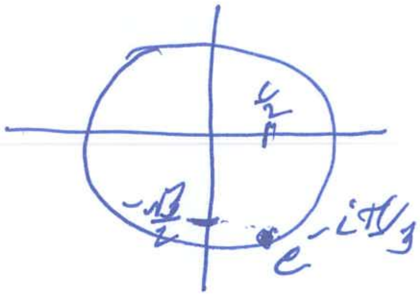


$$= \{z\in\mathbb{C}\mid z^4=1\}$$
$$= \{z\in\mathbb{C}\mid z^4-1=0\}$$
$$= \{z\in\mathbb{C}\mid (z-1)(z-i)(z+1)(z+i)=0\}$$

2.48 4th roots of $2-2i\sqrt{3}$

$2-2i\sqrt{3}$ has length $\sqrt{2^2+(-2\sqrt{3})^2}=\sqrt{4+4\cdot 3}=\sqrt{16}=4$

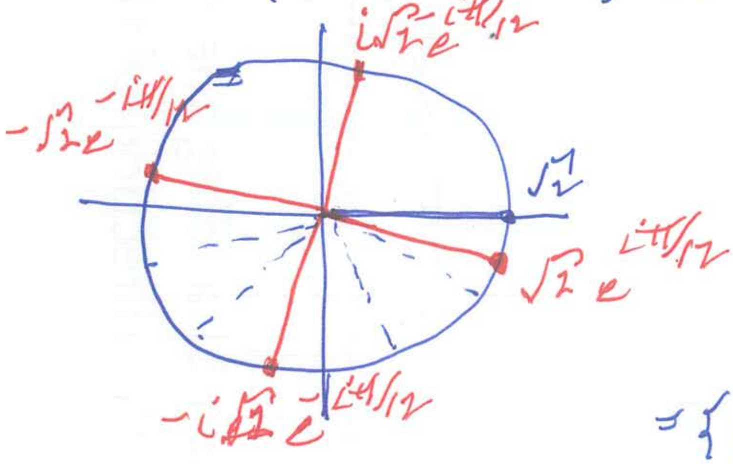
Then
$$2-2i\sqrt{3}=4\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$
$$=4e^{-i\pi/3}.$$



So $\{z\in\mathbb{C} \mid z=\sqrt[4]{2-2i\sqrt{3}}\}=\{z\in\mathbb{C}\mid z=(4e^{-i\pi/3})^{1/4}\}$

$=\{z\in\mathbb{C}\mid z=(4e^{-i\pi/3}\cdot 1)^{1/4}\}=\{z\in\mathbb{C}\mid z=\sqrt{2}e^{-i\pi/12}\cdot 1^{1/4}\}$

$=\{\sqrt{2}e^{i\pi/12}\cdot e^{i0\pi/4},\ \sqrt{2}e^{-i\pi/12}\cdot e^{i2\pi/4},\ \sqrt{2}e^{-i\pi/12}e^{i4\pi/4},\ \sqrt{2}e^{-i\pi/12}e^{i6\pi/4}\}$



$\{z\in\mathbb{C}\mid z=\sqrt[4]{2-2i\sqrt{3}}\}$

$=\{z\in\mathbb{C}\mid z^4=2-2i\sqrt{3}\}$

$=\{z\in\mathbb{C}\mid z^4-2+2\sqrt{3}i=0\}$

2.55 Factor $z^2-3iz-2$.

If $z^2-3iz-2=0$ then
$$z=\frac{-(-3i)\pm\sqrt{(-3i)^2-4\cdot 1\cdot(-2)}}{2}$$
$$=\frac{3i\pm\sqrt{-9+8}}{2}=\frac{3i\pm\sqrt{-1}}{2}$$
$$=\left\{\frac{3i+i}{2},\frac{3i-i}{2}\right\}=\{2i,i\}.$$

Then $(z-2i)(z-i)=0$

Check: $(z-2i)(z-i) = z^2 - iz - 2iz + 2i^2$
$= z^2 - 3iz - 2.$

So
$$\{z \in \mathbb{C} \mid z^2 - 3iz - 2 = 0\} = \{z \in \mathbb{C} \mid (z-2i)(z-i) = 0\}$$
$$= \{2i, i\}$$

2.59 Find the roots of $z^2 - 2z + 2$.

If $z^2 - 2z + 2 = 0$ then
$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 2}}{2}$$
$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$
$$= \left\{\frac{2+2i}{2}, \frac{2-2i}{2}\right\} = \{1+i, 1-i\}$$

So
$$\{z \in \mathbb{C} \mid z^2 - 2z + 2 = 0\} = \{z \in \mathbb{C} \mid (z-(1+i))(z-(1-i)) = 0\}$$
$$= \{1+i, 1-i\}.$$

2.60
$$\{z \in \mathbb{C} \mid z^3 - 3iz^2 - 2z = 0\}$$
$$= \{z \in \mathbb{C} \mid z(z^2 - 3iz - 2) = 0\}$$
$$= \{z \in \mathbb{C} \mid z(z-2i)(z-i) = 0\}$$
$$= \{z \in \mathbb{C} \mid (z-0)(z-2i)(z-i) = 0\}$$
$$= \{0, -2i, -i\}.$$