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Calculus Lect. 17
A. Ram

GRAPHING

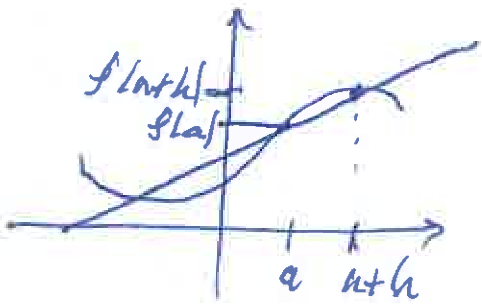
The fundamental theorem of change

$$\begin{pmatrix}\text{the evaluation} \\ \text{of } \frac{df}{dx} \text{ at } x=a\end{pmatrix} = \begin{pmatrix}\text{the slope of the graph} \\ \text{of } f(x) \text{ at } x=a\end{pmatrix}$$

In Math:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$\frac{f(a+h)-f(a)}{h} = \frac{\text{change in } f}{\text{change in } x}$$



$$= \frac{\text{rise}}{\text{run}} = \text{slope of line connecting } (a, f(a)) \text{ and } (a+h, f(a+h))$$

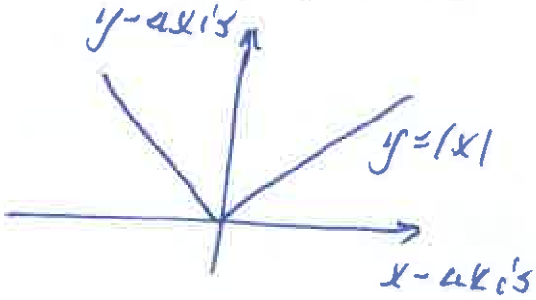
So
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \begin{pmatrix}\text{slope of } f \text{ at} \\ x=a\end{pmatrix}$$

A function $f: \mathbb{R} \to \mathbb{R}$ is <u>differentiable at $x=a$</u> if the slope of the graph of $f(x)$ at $x=a$ exists (is a real number).

Differentiability

Example $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$, where

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

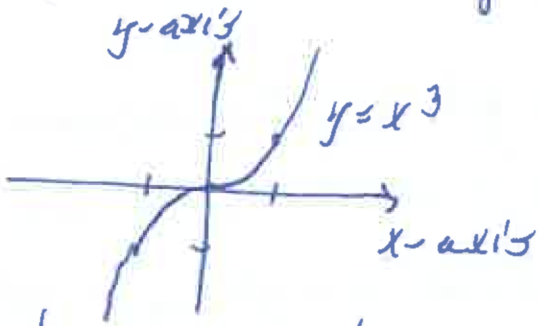


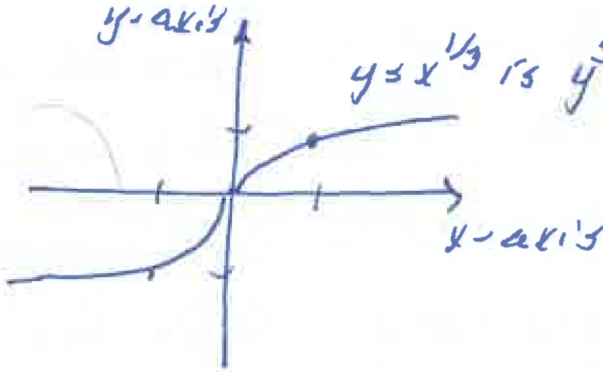
Then

$$f'(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0, \\ \text{does not exist}, & \text{if } x = 0. \end{cases}$$

So $f(x)$ is not differentiable at $x = 0$

Example $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^{1/3}$





Then $f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$

and $f'(0)$ is not a real number.

So $f(x)$ is not differentiable at $x = 0$.

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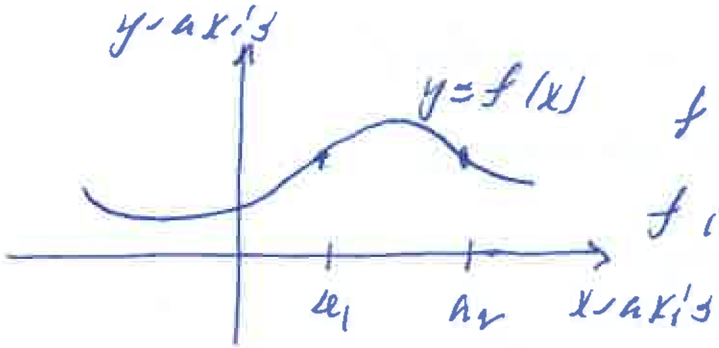
Increasing/decreasing

A function $f: \mathbb{R} \to \mathbb{R}$ is <u>increasing at $x=a$</u>
if it is going up at $x=a$
i.e. if $f(a+h) > f(a)$ for all small $h>0$
i.e. if the slope of $f(x)$ at $x=a$ is positive
i.e. if $f'(a) > 0$



f is increasing at $x=a_1$
f is decreasing at $x=a_2$

A function $f: \mathbb{R} \to \mathbb{R}$ is <u>decreasing at $x=a$</u>
if it is going down at $x=a$
i.e. if $f(a+h) < f(a)$ for all small $h>0$
i.e. if the slope of $f(x)$ at $x=a$ is negative
i.e. if $f'(a) < 0$.

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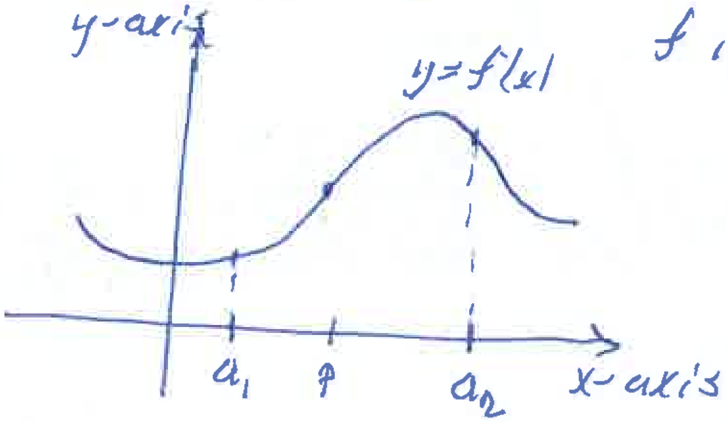
Concavity

A function $f:\mathbb{R}\to\mathbb{R}$ is <u>concave up at $x=a$</u>
if it is right side up bowl shaped at $x=a$
i.e. if the slope of f is getting larger
i.e. if $\frac{df}{dx}$ is increasing at $x=a$
i.e. if $f''(a)>0$



f is concave up at $x=a_1$

A function $f:\mathbb{R}\to\mathbb{R}$ is <u>concave down at $x=a$</u>
if it is upside down bowl shaped at $x=a$
i.e. if the slope of f is getting smaller
i.e. if $\frac{df}{dx}$ is decreasing at $x=a$
i.e. if $f''(a)<0$

A <u>point of inflection</u> is a point where
f changes from concave down to concave up
or from concave up to concave down.

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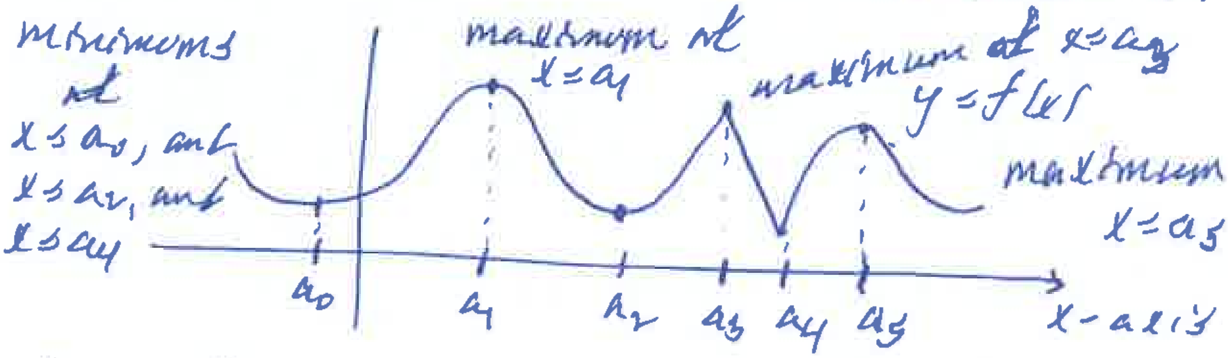
Maxima and minima

Let $f: \mathbb{R} \to \mathbb{R}$ be a function (continuous).



A <u>local maximum</u> of f is a point $x=a$, where $f(a)$ is bigger than the $f(x)$ around it.

A <u>local minimum</u> of f is a point $x=a$, where $f(a)$ is smaller than the $f(x)$ around it.

A <u>critical point</u>, ~~or stationary point~~, of f is a point where a maximum or minimum might occur.

<u>Note</u>: (1) If $f(x)$ is continuous and differentiable and $x=a$ is a maximum then

$$f'(a)=0 \quad \text{and} \quad f''(a)<0$$

(2) If $f(x)$ is continuous and differentiable and $x=a$ is a minimum then

$$f'(a)=0 \quad \text{and} \quad f''(a)>0.$$

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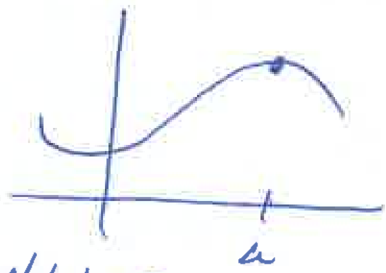
Stationary points

Where can a maximum or minimum occur?

(a) a point $x=a$ where $f(x)$ is differentiable and $f'(a)=0$

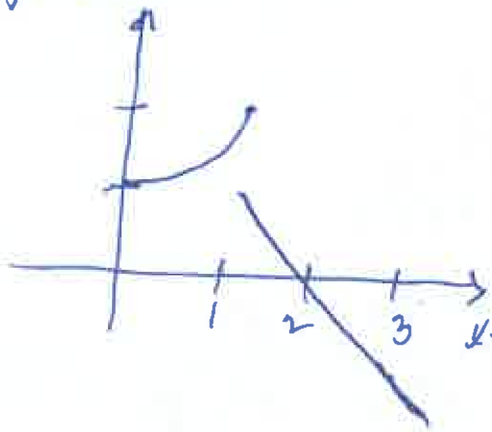
These points are stationary points



$f'(a)=0$ is a stationary point.

(b) A point $x=a$ where $f(x)$ is not continuous

(c) A point $x=a$ on the boundary of where $f(x)$ is defined.



$f: \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2+1, & \text{if } x \in \mathbb{R}_{[0,1]}, \\ 2-x, & \text{if } x \in \mathbb{R}_{>1}. \end{cases}$$

$x=0$ is a minimum

$x=1$ is a maximum