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Calculus test

A. Ram

Definite integrals

$\int_a^b f\,dx$ means $F(x)\Big]_{x=a}^{x=b}$ where $\frac{dF}{dx} = f$

$F(x)\Big]_{x=a}^{x=b}$ means $F(b) - F(a)$.

So $\int_a^b f\,dx = F(x)\Big]_{x=a}^{x=b} = F(b) - F(a)$

5.11
$$\int_1^{e^{\pi/2}} \frac{\sin(\log(x))}{x}\,dx = \int_1^{e^{\pi/2}} \sin(\log(x))\cdot\frac{1}{x}\,dx$$

$$= -\cos(\log(x))\Big]_{x=1}^{x=e^{\pi/2}} = -\cos(\log(e^{\pi/2})) - (-\cos(\log(1)))$$

$$= -\cos(\pi/2) + \cos(0) = -0 + 1 = 1,$$

since

$$\frac{d}{dx}(-\cos(\log(x))) = -(-\sin(\log(x))\cdot\frac{d}{dx}(\log(x)))$$

$$= \sin(\log(x))\cdot\frac{1}{x} = \frac{\sin(\log(x))}{x}.$$

<u>5.12</u> Assume $\int_0^1 f(x)\,dx = \frac{\pi}{2}$

This means $F(1)-F(0) = \frac{\pi}{2}$, where $\frac{dF}{dx} = f$.

Then

$$\int_1^e \frac{2f(\log(x))}{x}\,dx = \int_1^e 2f(\log(x))\cdot\frac{1}{x}\,dx$$

$$= 2F(\log(x))\Big]_{x=1}^{x=e} = 2F(\log(e)) - 2F(\log(1))$$

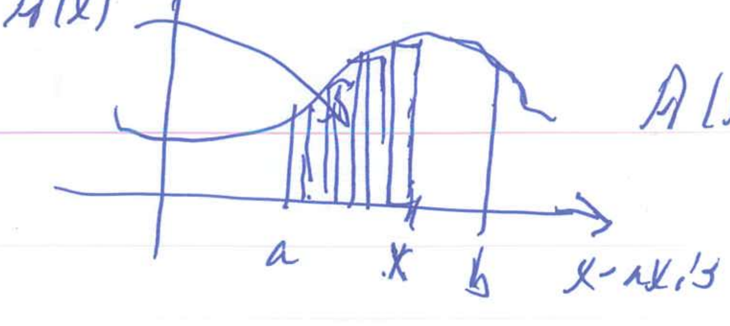
$$= 2F(1) - 2F(0) = 2(F(1)-F(0)) = 2\,\frac{\pi}{2} = \pi.$$

<u>Fundamental theorem of calculus</u>

You can add up a bunch of tiny things with a definite integral.

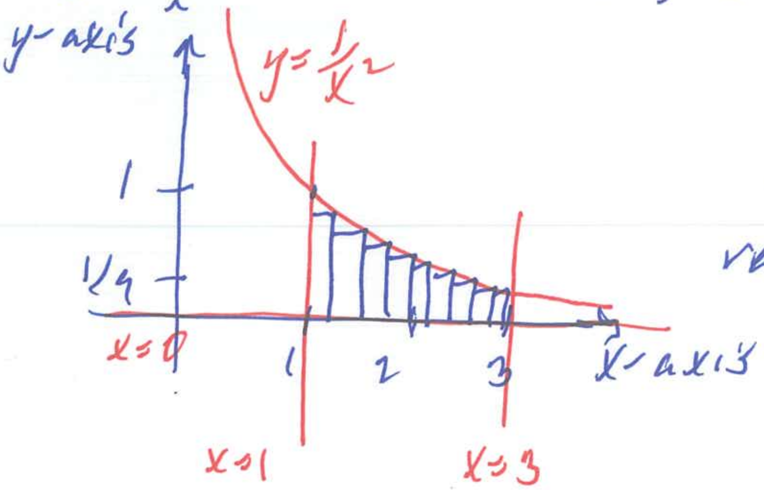
If $f:[a,b]\to\mathbb{R}_{\geq 0}$ is a differentiable function then

$$\int_a^b f(x)\,dx = \left(\begin{matrix}\text{Area under } f\\ \text{from } a \text{ to } b\end{matrix}\right) = A(b)-A(a)$$



$$A(x) = \lim_{\Delta x\to 0}\Big(\underbrace{f(a)\Delta x + \cdots + f(x-\Delta x)\Delta x}_{\text{add up areas of little rectangles.}}\Big)$$

5.3 Find the area enclosed by $y = \frac{1}{x^2}$, the x-axis, $x=1$ and $x=3$.



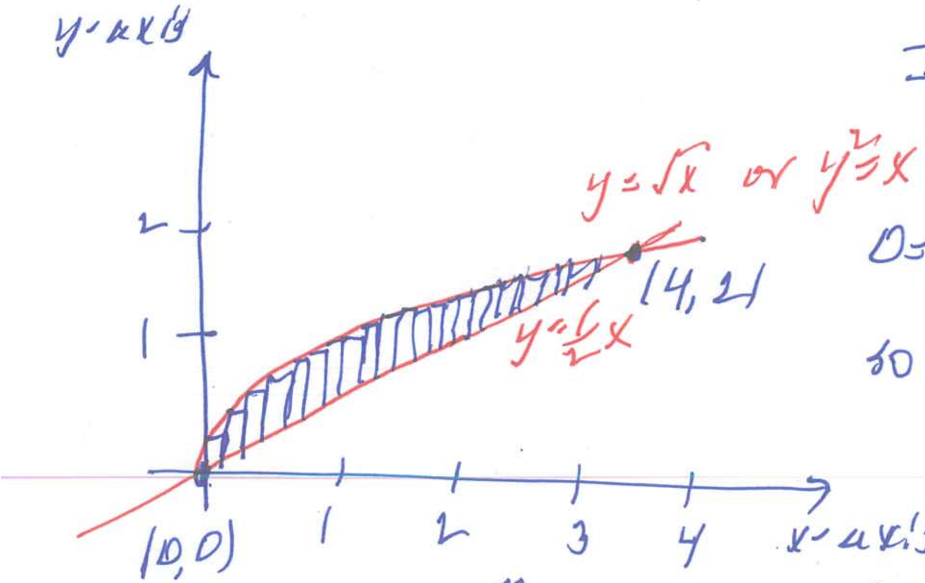
Add up little rectangles $\}\frac{1}{x^2}$, dx, from $x=1$ to $x=3$.

$$\int_{x=1}^{x=3} (y_{top} - y_{bottom})\,dx = \int_{x=1}^{x=3} \left(\frac{1}{x^2} - 0\right) dx = \int_{x=1}^{x=3} x^{-2}\,dx$$

$$= -x^{-1}\Big]_{x=1}^{x=3} = -3^{-1} - (-1^{-1}) = -\frac{1}{3} + \frac{1}{1} = \frac{2}{3}.$$

5.4 Find the area enclosed by

$y = \sqrt{x}$ and $y = \frac{1}{2}x$.



If $y^2 = x$ and $y = \frac{1}{2}x$ then $\left(\frac{1}{2}x\right)^2 = x$ and $0 = \frac{1}{4}x^2 - x = \left(\frac{1}{4}x - 1\right)x$ so that $x=0$ or $x=4$

Add up slices $\}y_{top} - y_{bottom}$, dx, from $x=0$ to $x=4$.

$$\int_{x=0}^{x=4} (y_{top} - y_{bottom})\,dx = \int_{x=0}^{x=4} (\sqrt{x} - \tfrac{1}{2}x)\,dx$$

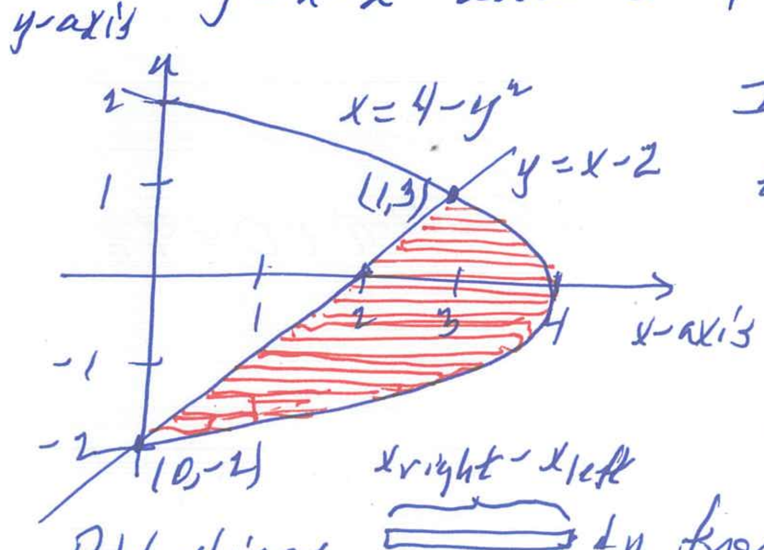
$$= \int_{x=0}^{x=4} (x^{1/2} - \tfrac{1}{2}x)\,dx = \tfrac{2}{3}x^{3/2} - \tfrac{1}{2}\cdot\tfrac{1}{2}x^2 \Big]_{x=0}^{x=4}$$

$$= \left(\tfrac{2}{3}\cdot 4^{3/2} - \tfrac{1}{4}\cdot 4^2\right) - \left(\tfrac{2}{3}\cdot 0^{3/2} - \tfrac{1}{4}\cdot 0^2\right) = \tfrac{2}{3}\cdot 2^3 - 4$$

$$= \tfrac{16}{3} - \tfrac{12}{3} = \tfrac{4}{3}.$$

<u>5.6</u> Find the area enclosed by

$y = x-2$ and $x = 4-y^2$



If $x=4-y^2$ and $y=x-2$ then

$y = 4-y^2-2 = 2-y^2$

So $y^2+y-2=0$.

So $(y+2)(y-1)=0$.

Add slices dy from $y=-2$ to $y=1$.

So $$\int_{y=-2}^{y=1} (x_{right} - x_{left})\,dy = \int_{y=-2}^{y=1} ((4-y^2) - (y+2))\,dy$$

$$= \int_{y=-2}^{y=1} (-y^2+4-y-2)\,dy = \int_{y=-2}^{y=1} (-y^2-y+2)\,dy$$

$$= -\tfrac{1}{3}y^3 - \tfrac{1}{2}y^2 + 2y\Big]_{y=-2}^{y=1} = \left(-\tfrac{1}{3}\cdot 1^3 - \tfrac{1}{2}\cdot 1^2 + 2\cdot 1\right) - \left(-\tfrac{1}{3}(-2)^3 - \tfrac{1}{2}(-2)^2 + 2(-2)\right)$$

$$= -\tfrac{1}{3} - \tfrac{1}{2} + 2 - \tfrac{8}{3} + 2 + 4 = 8 - 3 - \tfrac{1}{2} = 5 - \tfrac{1}{2} = \tfrac{9}{2}$$