620-295 Real analysis with applications Lecture 1, 27.07.2009

In the beginning...

humankind wanted to count. ... and

so we discovered...

<u>the set of positive integers</u>

$$\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$$

GREAT. But how do we talk about nothing?

... and so we discovered ...

<u>the set of nonnegative integers</u>

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$$

GREAT. But what if you go into debt :(

... and so we discovered the

<u>set of integers</u>

$$\mathbb{Z} = \{\cancel{0, 0, 0}, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Sets

A set is a collection of elements. Write

$$s \in S$$

if s is an element of the set S.

Let S and T be sets. The product of S and T is the set

$$S \times T = \{(s,t) \mid s \in S,\ t \in T\}$$

of pairs (s,t) with s an element of S and t an element of T.

Functions

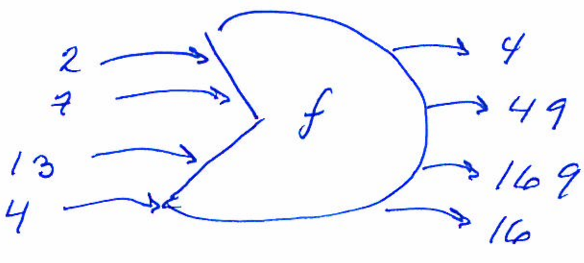
Let S and T be sets.

A function from S to T is an assignment

$$f : S \to T$$
$$s \mapsto f(s)$$

of $f(s) \in T$ to each element ~~of~~ $s \in S$.

Example



$$x^2 : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$$
$$n \mapsto n^2$$

What does 5 mean, really?

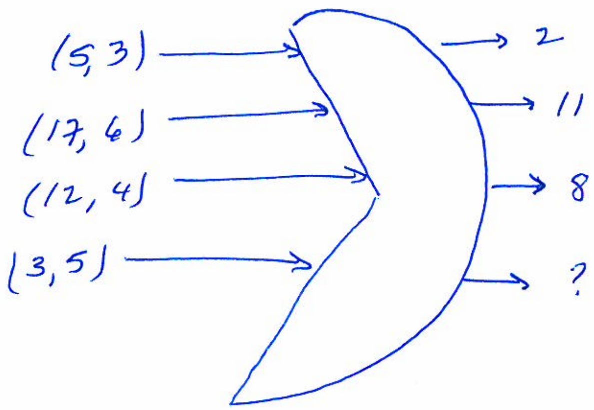
$$5 = 1+1+1+1+1$$

$\mathbb{Z}_{>0}$ is not just a set. It is a set with an operation.

Let S be a set.

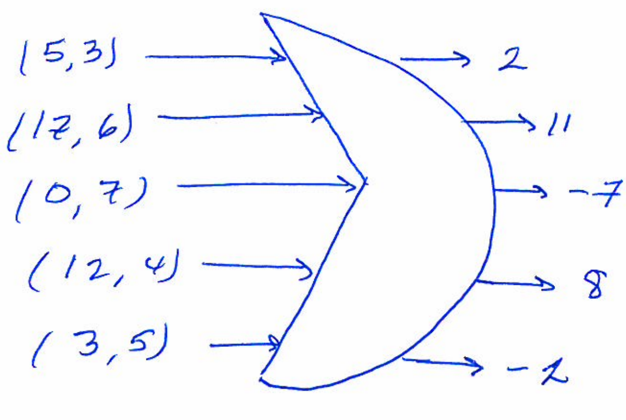
An <u>operation</u> <u>on</u> <u>S</u> is a function $\diamond : S \times S \to S$.

<u>Example</u> $- : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$



I LIED. $- : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is not <u>well</u> <u>defined</u>.

$- : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. is better



Let S be a set and let $\diamond : S \times S \to S$ be an operation on S.

The operation $\diamond : S\times S \to S$ is <u>commutative</u> if it satisfies:

if $s_1, s_2 \in S$ then $s_1 \diamond s_2 = s_2 \diamond s_1$

The operation $\diamond : S \times S \to S$ is <u>associative</u> if it satisfies:

if $s_1, s_2, s_3 \in S$ then $s_1 \diamond (s_2 \diamond s_3) = s_1 \diamond (s_2 \diamond s_3)$

A <u>commutative</u> <u>monoid</u> <u>without</u> <u>identity</u> is a set S with an operation that is commutative and associative.

<u>Example</u> $\mathbb{Z}_{>0}$ with the operation
$$\begin{array}{rl} + : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} & \to \mathbb{Z}_{>0} \\ (s,t) & \mapsto s+t \end{array}$$
is a commutative monoid without identity.

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<u>Example</u> $\mathbb{Z}$ with the operation $-: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$
$$(s,t) \mapsto s-t$$

The operation $-: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ is not commutative since
$$5-3 \neq 3-5.$$
The operation $-: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ is not associative since
$$5-(3-0) \neq (5-3)-0$$

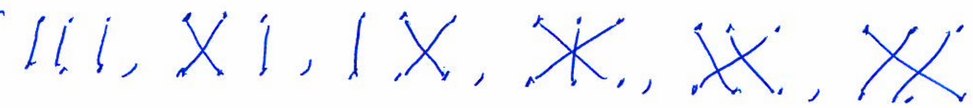
<u>Example</u> The clock monoid:

$$\mathcal{C} = \{12, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \quad \text{with} \quad \begin{array}{l} \diamond: \mathcal{C}\times\mathcal{C} \to \mathcal{C} \\ (3,6) \mapsto 9 \end{array}$$

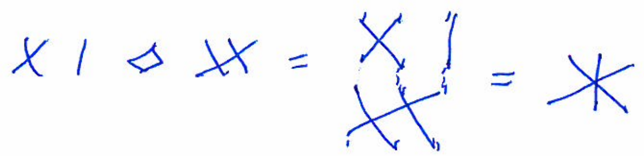
$$3 \diamond 6 = 9, \quad 6 \diamond 7 = 1, \quad 10 \diamond 8 = 6, \quad 9 \diamond 10 = 7$$

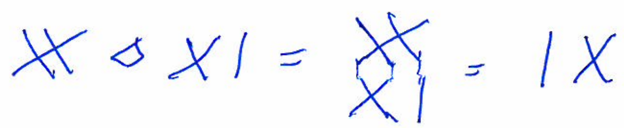
The set $\mathcal{C}$ with operation $\diamond: \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ given by clock addition is a commutative monoid without identity.

<u>Example</u> The connect the dots monoid

$S_3 = \{$  $\}$

(3 dots on top, 3 dots on bottom, top dots connected to bottom). Define an operation on S_3:





S_3 with operation $\circ : S_3 \times S_3 \to S_3$ is not a commutative monoid without identity.