The rational numbers

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ with operations

$$+ : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z} \qquad \text{and} \qquad \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$$
$$(s,t) \mapsto s+t \qquad\qquad (s,t) \mapsto st$$

is good until... what if you only want part of the sausage?... and so we discovered...

The rational numbers is the set

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z},\ b \neq 0 \right\}$$

with $\quad \dfrac{a}{b} = \dfrac{c}{d}, \quad$ if $\; ad = bc$

and operations

$$+ : \mathbb{Q}\times\mathbb{Q} \to \mathbb{Q} \quad \text{and} \quad \cdot : \mathbb{Q}\times\mathbb{Q} \to \mathbb{Q}$$

given by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$$

Example: $\dfrac{583951}{911} \stackrel{?}{=} 641$ How do you know?

Well
$$\begin{array}{r} 641 \\ 911 \\ \hline 641 \\ 641 \\ 5769 \\ \hline 583951 \end{array}$$

Example: $\dfrac{1-x^7}{1-x} \stackrel{?}{=} 1+x+x^2+x^3+x^4+x^5+x^6$

HOW DO YOU KNOW? Well

$$(1-x)(1+x+x^2+x^3+x^4+x^5+x^6) = 1+x+x^2+x^3+x^4+x^5+x^6$$
$$-x-x^2-x^3-x^4-x^5-x^6-x^7$$
$$= 1-x^7.$$

Example $\dfrac{1}{1-x} \stackrel{?}{=} 1+x+x^2+x^3+\dots$

$$(1-x)(1+x+x^2+x^3+\dots) = 1+x+x^2+x^3+x^4+\dots$$
$$-x-x^2-x^3-x^4-\dots$$
$$= 1.$$

Example Show that if $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ then $\frac{a}{b}+\frac{c}{d} = \frac{c}{d}+\frac{a}{b}$.

Proof: Assume $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$.

To show: $\frac{a}{b}+\frac{c}{d} = \frac{c}{d}+\frac{a}{b}$.

$$LHS = \frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd}$$

$$RHS = \frac{c}{d}+\frac{a}{b} = \frac{cb+ad}{db}.$$

Then $RHS = \dfrac{cb+ad}{bd}$ since $\begin{matrix}\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z} \\ (s,t)\mapsto st\end{matrix}$ is commutative

Then $RHS = \dfrac{ad+cb}{bd}$ since $+ : \begin{matrix}\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z} \\ (s,t)\mapsto s+t\end{matrix}$ is commutative

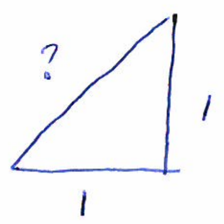
Then $RHS = \dfrac{ad+bc}{bc}$ since $\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$, $(s,t)\mapsto st$ is commutative

So $RHS = LHS$.

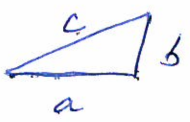
So $\dfrac{a}{b}+\dfrac{c}{d} = \dfrac{c}{d}+\dfrac{a}{b}$.

<u>The real numbers</u>

The number system $\mathbb{Q}$ with ~~$f: \mathbb{Q}\times\mathbb{Q}\to\mathbb{Q}$~~ ~~and f:~~ operations addition and multiplication is great... until...

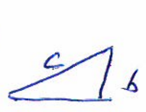


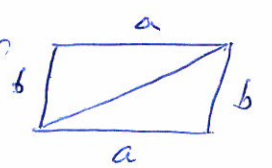
<u>The Pythagorean Theorem</u>

Let 

be a right triangle with leg lengths a and b and with hypotenuse length c.

Then $a^2+b^2=c^2$.

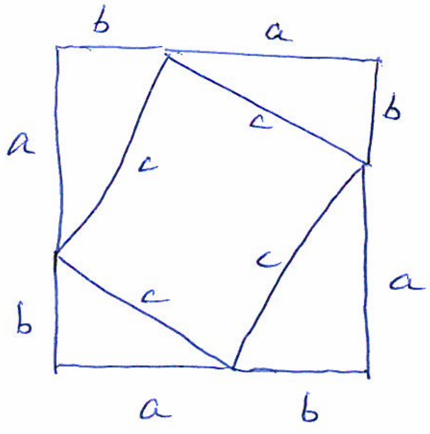
<u>Proof</u> Area of 

$= \frac{1}{2}$ (Area of 

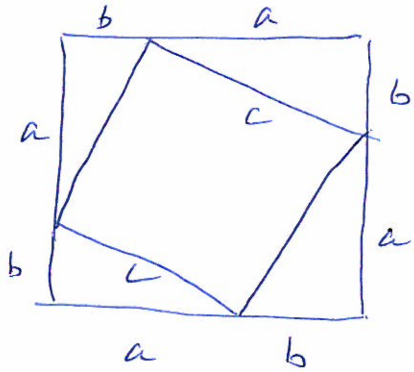
)

$= \frac{1}{2}ab$.

Since

Area of 

$= 4(\frac{1}{2}ab) + c^2 = 2ab + c^2$, and

Area of 

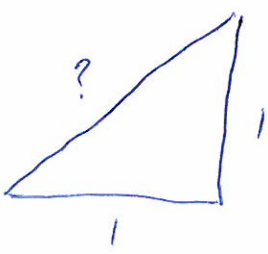
$= (a+b)^2$,

$$2ab + c^2 = (a+b)^2.$$

So $\quad 2ab + c^2 = a^2 + 2ab + b^2$

So $\quad c^2 = a^2 + b^2$. //

So

if 

then $?^2 = 2$.

<u>Theorem</u> There does not exist $x \in \mathbb{Q}$ with $x^2 = 2$.

Proof Proof by contradiction

Assume that there exists $x \in \mathbb{Q}$ with $x^2 = 2$.

Then $x = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, $b \neq 0$, such that a and b have no common factors.

Then $2 = x^2 = \frac{a^2}{b^2}$.

So $2 \cdot b^2 = a^2$.

So 2 divides a.

So 4 divides a^2.

So 4 divides $2b^2$.

So 2 divides b.

So 2 is a common factor of a and b.

Contradiction.

So $x \neq \frac{a}{b}$ with $a, b \in \mathbb{Z}$, $b \neq 0$ with a and b having no common factors.

So there does not exist $x \in \mathbb{Q}$ with $x^2 = 2$.

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