<u>620-295 Real Analysis with applications</u>, Lect. 5, 04.08.2009 (1)

Let S and T be sets.

A <u>function</u> $f: S \to T$ is given by associating to each element $s \in S$ a unique element $f(s) \in T$.

$$\begin{aligned} f: S &\to T \\ s &\mapsto f(s) \end{aligned}$$

To check whether an attempted function $f: S \to T$ is <u>well defined</u> one must check

(a) if $s \in S$ then $f(s) \in T$,

(b) if $s_1, s_2 \in S$ and $s_1 = s_2$ then $f(s_1) = f(s_2)$

A function is <u>injective</u> if it satisfies

if $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$ then $s_1 = s_2$

A function $f: S \to T$ is <u>surjective</u> if it satisfies

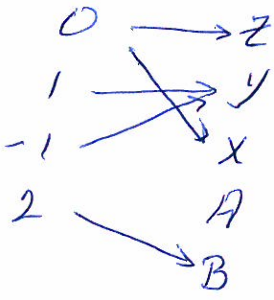
if $t \in T$ then there exists $s \in S$ such that $f(s) = t$.

A function $f: S \to T$ is <u>bijective</u> if it ~~satisfies~~ is both injective and surjective.

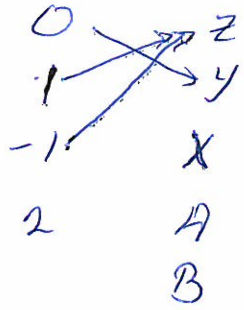
The <u>graph</u> of a function $f: S \to T$ is the set

$$\Gamma_f = \{(s,t) \in S \times T \mid t = f(s)\}.$$

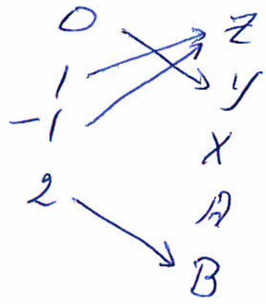
Examples $S = \{0, 1, -1, 2\}$ $T = \{z, y, x, A, B\}$.



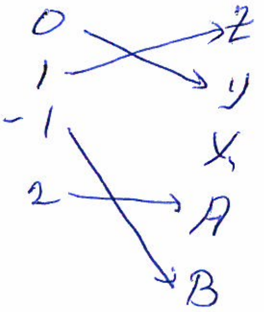
not a function



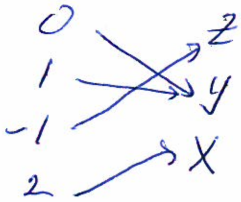
not a function



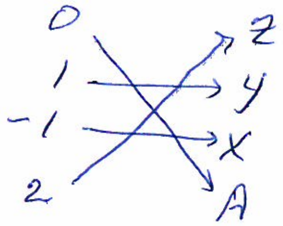
function



injective
not surjective



surjective
not injective



bijective.

Let $f: S \to T$ and $g: T \to U$ be functions

The composition of f and g is the function $g \circ f: S \to T$ given by

$$(g \circ f)(s) = g(f(s)), \quad \text{for } s \in S.$$

Let S be a set. The identity function on S is the function

$$\begin{aligned} id_S : S &\to S \\ s &\mapsto s. \end{aligned}$$

Let $f : S \to T$ be a function

An inverse function to f is a function $f^{-1} : T \to S$ such that

$$f \circ f^{-1} = id_T \quad \text{and} \quad f^{-1} \circ f = id_S .$$

Examples In Calculus I

(a) x is the identity function

(b) $\sqrt{x}$ is the inverse function to x^2,

$$\sqrt{x^2} = x \quad \text{and} \quad (\sqrt{x})^2 = x,$$

i.e. $\sqrt{x}$ is the function that undoes x^2.

(c) $\log x$ is the inverse function to e^x

$$\log(e^x) = x \quad \text{and} \quad e^{\log x} = x$$

i.e. $\log x$ is the function that undoes e^x.

(d) $\sin^{-1} x$ is the function that undoes $\sin x$

$$\sin^{-1}(\sin x) = x \quad \text{and} \quad \sin(\sin^{-1} x) = x$$

i.e. $\sin^{-1} x$ is the inverse function to $\sin x$.

(e) $\tan^{-1} x$ is the function that undoes $\tan x$

$$\tan^{-1}(\tan x) = x \quad \text{and} \quad \tan(\tan^{-1} x) = x$$

i.e. $\tan^{-1} x$ is the inverse function to $\tan x$.

<u>Theorem</u> Let $f: S \to T$ be a function.
The inverse function to f exists if and only if f is bijective.

<u>Proof</u> $\Rightarrow$

To show: If the inverse function to f exists then f is bijective.

Assume the inverse function to f exists.

To show: f is bijective.

To show: (a) f is injective
(b) f is surjective.

(a) To show: If $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$ then $s_1 = s_2$

Assume $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$

To show: $s_1 = s_2$

Since $f^{-1}: T \to S$ exists,

$$s_1 = f^{-1}(f(s_1)) = f^{-1}(f(s_2)) = s_2.$$

So f is injective.

(b) To show: f is surjective

To show: If $t \in T$ then there exists $s \in S$ such that $f(s) = t$.

Assume $t \in T$.

Define $s = f^{-1}(t)$.

To show: $f(s) = t$.

$$f(s) = f(f^{-1}(t)) = t,$$

since f^{-1} is the inverse function to f.

So f is surjective

So f is bijective.

$\Leftarrow$: To show: If f is bijective then the inverse function to f exists.

Assume f is bijective

To show: $f^{-1}: T \to S$ exists.

Let $t \in T$. Define

$$f^{-1}(t) = s, \quad \text{where } s \in S \text{ such that } f(s) = t.$$

This s exists since f is surjective.

To show: $f^{-1}: T \to S$ is well defined.

To show: (a) If $t \in T$ then $f^{-1}(t) \in S$

(b) If $t_1, t_2 \in T$ and $t_1 = t_2$ then $f^{-1}(t_1) = f^{-1}(t_2)$

(a) Assume $t \in T$.

By the definition of f^{-1}, $f^{-1}(t) = s$ with $s \in S$.

(b) Assume $t_1, t_2 \in T$ and $t_1 = t_2$.

$f^{-1}(t_1) = s_1$, with $s_1 \in S$ and $f(s_1) = t_1$

$f^{-1}(t_2) = s_2$, with $s_2 \in S$ and $f(s_2) = t_2$

To show: $f^{-1}(t_1) = f^{-1}(t_2)$.

Since f is injective and

$$f(s_1) = t_1 = t_2 = f(s_2),$$

then $s_1 = s_2$.

So $f^{-1}(t_1) = f^{-1}(t_2)$

So f^{-1} is well defined.

So $f^{-1}: S \to T$ exists if and only if $f: S \to T$ is bijective. //.