620-295 Real Analysis with applications, lect 6, 06/08/2009 ①

Polynomials

$$\mathbb{Q}[x] = \left\{ a_0 + a_1 x + a_2 x^2 + \cdots \;\middle|\; \begin{array}{l} a_i \in \mathbb{Q}, \text{ and all but a finite} \\ \text{number of the } a_i \text{ are } 0 \end{array} \right\}$$

$$\mathbb{Q}[[x]] = \{ a_0 + a_1 x + a_2 x^2 + \cdots \mid a_i \in \mathbb{Q} \}.$$

Examples $1+x = 1+x+0x^2+0x^3+\cdots \in \mathbb{Q}[x]$

(and in $\mathbb{Q}[[x]]$).

$1+x+x^2+x^3+\cdots \in \mathbb{Q}[[x]]$.

$1+2x+3x^2+4x^3+\cdots \in \mathbb{Q}[[x]]$.

$\mathbb{Q}[x]$ and $\mathbb{Q}[[x]]$ have operations:

Addition $(3+2x+7x^2)(5+3x) = 15+10x+35x^2$

$$+ 9x + 6x^2 + 21x^3$$

$$= 15+19x+41x^2+21x^3$$

$(3+2x+7x^2)+(5+3x) = 8+5x+7x^2.$

Multiplication

$(1+x+x^2+x^3+\cdots)+(1+2x+3x^2+4x^3+\cdots)$

$$= 2+3x+4x^2+5x^3+\cdots$$

$(1+x+x^2+x^3+\cdots)(1+2x+3x^2+4x^3+\cdots)$

$= 1+(2+1)x+(3+2+1)x^2+(4+3+2+1)x^3+\cdots$

$= 1+3x+6x^2+10x^3+15x^4+\cdots$

$\mathbb{Q}[x]$ and $\mathbb{Q}[[x]]$ are rings.

$$\mathbb{Q}(x) = \left\{ \frac{a(x)}{b(x)} \;\middle|\; a(x), b(x) \in \mathbb{Q}[x], \quad b(x) \neq 0 \right\}$$

with $$\frac{a(x)}{b(x)} = \frac{c(x)}{d(x)} \quad \text{if} \quad a(x)d(x) = b(x)c(x).$$

$$\mathbb{Q}((x)) = \left\{ \frac{a(x)}{b(x)} \;\middle|\; a(x), b(x) \in \mathbb{Q}[[x]], \; b(x) \neq 0 \right\}$$

with $$\frac{a(x)}{b(x)} = \frac{c(x)}{d(x)} \quad \text{if} \quad a(x)d(x) = b(x)c(x).$$

$\mathbb{Q}(x)$ and $\mathbb{Q}((x))$ have operations given by

$$\frac{a(x)}{b(x)} + \frac{c(x)}{d(x)} = \frac{a(x)d(x) + b(x)c(x)}{b(x)d(x)}, \quad \text{and}$$

$$\frac{a(x)}{b(x)} \frac{c(x)}{d(x)} = \frac{a(x)c(x)}{b(x)d(x)}.$$

<u>Examples</u> $$\frac{3+2x^2+x^3}{1-x^4} \in \mathbb{Q}(x)$$

$$\frac{1+x+x^2+x^3+\cdots}{1+2x+3x^2+4x^3+\cdots} \in \mathbb{Q}((x)).$$

$\mathbb{Q}(x)$ and $\mathbb{Q}((x))$ are fields.

Particularly useful examples of elements of $\mathbb{Q}[[x]]$

(1) $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \in \mathbb{Q}[[x]].$

(2) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \cdots \quad \in \mathbb{Q}[[x]].$

Here

$$k! = k(k-1)(k-2)\cdots 3\cdot 2\cdot 1, \quad \text{for } k \in \mathbb{Z}_{>0}.$$

For example: $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120.$

(3) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \cdots \quad \in \mathbb{Q}[[x]]$

(4) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots \quad \in \mathbb{Q}[[x]].$

(5) $\tan x = \dfrac{\sin x}{\cos x} \in \mathbb{Q}((x)).$

(6) $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots \quad \in \mathbb{Q}[[x]]$

(7) $\log(1+x) = \int\left(\dfrac{1}{1+x}\right)dx = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \cdots \quad \in \mathbb{Q}[[x]].$

(8) $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \quad \in \mathbb{Q}[[x]]$

(9) $\tan^{-1} x = \int \dfrac{1}{1+x^2}\,dx = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots \quad \in \mathbb{Q}[[x]].$

The derivative with respect to x is a function

$$\frac{d}{dx} : \mathbb{Q}[[x]] \longrightarrow \mathbb{Q}[[x]]$$

such that if $a(x), b(x) \in \mathbb{Q}[[x]]$ then

(A) $\frac{d}{dx}(x) = 1$

(B) $\frac{d}{dx}(a(x)+b(x)) = \frac{d}{dx}(a(x)) + \frac{d}{dx}(b(x))$

(C) $\frac{d}{dx}(c\,a(x)) = c \cdot \frac{d}{dx}(a(x))$, if $c \in \mathbb{Q}$

(D) $\frac{d}{dx}(a(x)\,b(x)) = a(x) \cdot \frac{d}{dx}(b(x)) + \frac{d}{dx}(a(x)) \cdot b(x)$.

HW: Using only the properties (A), (B), (C), (D) show that

$$\frac{d}{dx}(x^{6284}) = 6284\, x^{6283}.$$

Theorem Let

$$a(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \in \mathbb{Q}[[x]].$$

Then

$$a_k = \left(\frac{1}{k!}\right) \left(\underbrace{\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx} \cdots \left(\frac{d}{dx}\right.\right.\right.}_{k \text{ times}} (a(x))\cdots)))\right)\Bigg|_{x=0}$$

Write

$$\frac{d^k a}{dx^k} \quad \text{for} \quad \underbrace{\frac{d}{dx}\left(\frac{d}{dx}\left(\cdots\left(\frac{d}{dx}\right.\right.\right.}_{k \text{ times.}} (a(x))\cdots)))$$

Proof Case 1 $k=0$.

$$a(x)\Big|_{x=0} = a_0 + a_1\cdot 0 + a_2\cdot 0^2 + \cdots = a_0$$

Case 2: $k=1$

$$\frac{d}{dx}(a(x)) = 0 + a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$$

So $$\frac{d}{dx}(a(x))\Big|_{x=0} = a_1 + 2a_2\cdot 0 + 3a_3\cdot 0^2 + 4a_4\cdot 0^3 + \cdots = a_1$$

Case 3 $k=2$.

$$\frac{d^2a}{dx^2} = 0 + 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + \cdots$$

$$\frac{d^2a}{dx^2}\Big|_{x=0} = 2a_2 + 3\cdot 2a_3\cdot 0 + 4\cdot 3a_4\cdot 0^2 + 5\cdot 4a_5\cdot 0^3 + \cdots = 2a_2$$

So $$a_2 = \frac{1}{2}\,\frac{d^2a}{dx^2}\Big|_{x=0}.$$

Case 4 $k=3$.

$$\frac{d^3a}{dx^3} = 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4x + 5\cdot 4\cdot 3a_5x^2 + 6\cdot 5\cdot 4a_6x^3 + \cdots$$

$$\frac{d^3a}{dx^3}\Big|_{x=0} = 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4\cdot 0 + 5\cdot 4\cdot 3a_5\cdot 0^2 + \cdots = 3\cdot 2a_3$$

So $$a_3 = \frac{1}{3\cdot 2}\cdot\frac{d^3a}{dx^3}\Big|_{x=0}$$