620-295 Real Analysis with applications lecture 9, 13 August 2009 ①

Let S be a set.

A relation on S is a subset Γ of $S\times S$

Write $x\le y$ if $(x,y)\in\Gamma$.

Example $S=\{\alpha,\beta,\gamma\}$

$$S\times S=\left\{\begin{matrix}(\alpha,\alpha), (\alpha,\beta), (\alpha,\gamma)\\ (\beta,\alpha)\ (\beta,\beta)\ (\beta,\gamma)\\ (\gamma,\alpha)\ (\gamma,\beta)\ (\gamma,\gamma)\end{matrix}\right\} \qquad \Gamma=\{(\beta,\beta), (\beta,\gamma), (\gamma,\alpha)\}.$$

Then $\beta\le\beta$, $\beta\le\gamma$, $\gamma\le\alpha$ but $\alpha\not\le\alpha$, $\beta\not\le\alpha$.

A partial order on S is a relation $\le$ on S such that

(a) If $x,y,z\in S$ and $x\le y$ and $y\le z$ then $x\le z$.

(b) If $x,y\in S$ and $x\le y$ and $y\le x$ then $x=y$.

A total order on S is a relation $\le$ on S such that

(a) If $x,y\in S$ and $x\le y$ and $y\le z$ then $x\le z$

(b) If $x,y\in S$ and $x\le y$ and $y\le x$ then $x=y$

(c) If $x,y\in S$ then $x\le y$ or $y\le x$.

<u>Example</u> (partial order that is not a total order).

Let S be a set.

The <u>power set</u> of S is the set 2^S of all subsets of S.

Define a partial <u>order</u> on 2^S by

$X \leq Y$ if X is a subset of Y.

For example: If

$S = \{\alpha, \beta, \gamma\}$ then

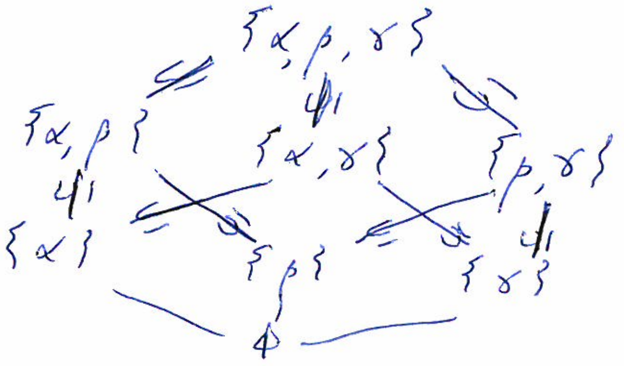
$$2^S = \left\{ \begin{matrix} \{\alpha,\beta\}, & \{\alpha\}, & \{\gamma\} \\ \{\alpha,\gamma\}, & \phi, & \{\beta,\gamma\} \\ \{\alpha,\beta,\gamma\}, & \{\beta\} & \end{matrix} \right\}$$ or we could write

$2^S = \{Z, Y, X, W, V, U, T, S\}$ where

$Z = \{\alpha\}$, $Y = \{\beta\}$, $X = \{\gamma\}$

$W = \{\alpha,\beta\}$, $V = \{\alpha,\gamma\}$, $U = \{\beta,\gamma\}$ and $T = \phi$.

Then



Note that:

$\{\alpha,\beta\} \not\leq \{\alpha,\gamma\}$

and $\{\alpha,\gamma\} \not\leq \{\alpha,\beta\}$.

Examples of total orders

(1) Define an order on $\mathbb{Z}_{>0}$ by $x \le y$ if

(a) $x = y$, or

(b) there exists $n \in \mathbb{Z}_{>0}$ with $x + n = y$.

(2) Define an order on $\mathbb{Z}_{\geq 0}$ by $x \le y$ if

there exists $n \in \mathbb{Z}_{\geq 0}$ with $x + n = y$.

(3) Define an order on $\mathbb{Z}$ by

$x \le y$ if $y + (-x) \in \mathbb{Z}_{\geq 0}$.

(4) Define an order on $\mathbb{Q}$ by

$x \le y$ if $y + (-x) \in \mathbb{Q}_{\geq 0}$,

where $\mathbb{Q}_{\geq 0} = \{ \frac{a}{b} \mid a, b \in \mathbb{Z}_{\geq 0}, b \neq 0 \}$

(5) Define an order on $\mathbb{R}$ by

$x \le y$ if $y + (-x) \in \mathbb{R}_{\geq 0}$,

where $\mathbb{R}_{\geq 0} = \{ a_0 . a_1 a_2 a_3 \ldots \mid a_0 \in \mathbb{Z}_{\geq 0}, a_1, a_2, \ldots \in \{0, \ldots, 9\} \}$

(6) There is no favourite order on $\mathbb{C}$.

An <u>ordered</u> <u>monoid</u> <u>possibly</u> <u>without</u> <u>identity</u> is a commutative monoid possibly without identity $(S,+)$ with a partial order $\le$ such that

if $x,y,z \in S$ and $x \le y$ then $x+z \le y+z$.

An <u>ordered</u> <u>monoid</u> is a commutative monoid $(S,+)$ with a partial order $\le$ such that

if $x,y,z \in S$ and $x \le y$ then $x+z \le y+z$

An <u>ordered</u> <u>group</u> is an abelian group $(S,+)$ with a partial order $\le$ such that

if $x,y,z \in S$ and $x \le y$ then $x+z \le y+z$.

An <u>ordered</u> <u>ring</u> is a commutative ring $(S,+,\cdot)$ with a partial order. $\le$ such that

(a) if $x,y,z \in S$ and $x \le y$ then $x+z \le y+z$,

(b) if $x,y \in S$ and $x \ge 0$ and $y \ge 0$ then $xy \ge 0$.

An <u>ordered</u> <u>field</u> is a field $(S,+,\cdot)$ with a total order $\le$ such that

(a) if $x,y,z \in S$ and $x \le y$ then $x+z \le y+z$

(b) if $x,y \in S$ and $x \ge 0$ and $y \ge 0$ then $xy \ge 0$

<u>Example</u> Another important order on $\mathbb{Z}_{\geq 0}$.

Define

$x \mid y$ if (a) there exists $n \in \mathbb{Z}_{\geq 0}$ such that $xn = y$, or

(b) $x = y$.

Note that $(\mathbb{Z}_{\geq 0}, \cdot)$ is a commutative monoid.

and $(\mathbb{Z}_{\geq 0}, \cdot)$ and the partial order $\mid$ is and ordered monoid.

<u>Example</u> Another important order on $\mathbb{Z}$.

Define $x \mid y$ if

(a) $x = y$, or

(b) there exists $n \in \mathbb{Z}$ such that $xn = y$.

Then $(\mathbb{Z}, \cdot)$ is a commutative monoid and $(\mathbb{Z}, \cdot)$ with the partial order $\mid$ is an ordered monoid.