Department of Mathematics and Statistics
620–295
Real analysis with applications

Laboratory Class 2: Iterative methods for solving nonlinear equations

Before starting, copy the folder `Lab2` from the lab server `M&S Lab Materials\620-295` to `D:MATLAB` and set the path to `D:MATLAB` including subfolders.

1 Picard iteration

The simplest iterative method to find the root x of a function F

$$F(x) = 0 \tag{1}$$

is to rewrite Eq. 1 in the *fixed point form*

$$x = f(x) \tag{2}$$

which can be done in infinitely many ways, and seek a solution through the *Picard iteration*

$$a_{n+1} = f(a_n)$$

for $n = 0, 1, 2, 3, ...$, starting from some initial guess a_0.

1.0.1 Exercise

Evaluate $x^3 + 4x^2 - 10$ at $x = 1$ and $x = 2$ and use this to conclude that the equation

$$x^3 + 4x^2 - 10 = 0 \tag{3}$$

must have a root in the interval [1,2]. Show that the equation $x^3 + 4x - 10 = 0$ is equivalent to the equations

$$x = x - x^3 - 4x^2 + 10 = \mathrm{g(x)}, \quad \text{and} \tag{4}$$

$$x = \sqrt{\frac{10}{4+x}} = \mathrm{h(x)}. \tag{5}$$

1.1 Self-maps

A sufficient condition for Eq. 2 (and hence Eq. 1) to have a solution is for the function f to be a *self-map* i.e. f must map some interval $[a, b]$ into itself (or an interval contained in $[a, b]$). One way to see this is to plot f.

The easiest way to plot a function in Matlab is to use `fplot` e.g.

```
>>g = @(x) x-x.^3 -4*x.^2 +10;  % use array operators in anonymous function
>>fplot(g, [1 2])
```

1.1.1 Exercise

Try plotting the functions g, h to ascertain which of $g(x), h(x)$ is a self-map for the interval [1,2].

1.2 Cobweb diagrams

A geometric view of Picard iteration is given by a *cobweb diagram*.

1.2.1 Exercise

Using the M-file `cobweb2.m` from the `Lab2` folder, generate cobweb diagrams for the Picard iterations for Eqs. 4 and 5 above. e.g

```
>>g = @(x) x-x.^3 -4*x.^2 +10;  % use array operators in anonymous function
>>cobweb2(g,1.5,10)   % no semicolon
```

which uses a starting value $a_0 = 1.5$ and does 10 iterates. Try a variety of starting guesses a_0 and number of iterates to see how the sequence of Picard iterates behaves.

The plots show both the sequence a_n of iterates, plotted versus n, as well as the cobweb diagram. You can resize the plot by pulling on the bottom right hand corner.

1.3 Contractive sequences of iterates

To be useful, we want the sequence of Picard iterates to converge to the fixed point. A sufficient condition for convergence is that the function f be *contractive* over the interval i.e.

$$|f(x) - f(y)| < \alpha|x - y|$$

where the *contractive constant* $\alpha < 1$.

1.3.1 Exercise

For which cases above do you think the function f is contractive? What feature of the graph of f might tell you it is contractive? Try Picard iterations (using `cobweb2`) to solve the equations (also equivalent to Eq. 3)

$$x = \sqrt{10 - x^3}/2 \tag{6}$$

$$x = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \tag{7}$$

Which do you think has the smallest value of α?

1.4 Towards chaos

Now consider the Picard iteration

$$a_{n+1} = \lambda a_n^2(1 - a_n)$$

with real parameter $0 \leq \lambda \leq 6.75$, defined for $x \in [0, 1]$.

1.4.1 Exercise

Using `cobweb2`, explore iterates of the map for the following parameter values:

$$\lambda = 4, 5, 5.5, 5.7, 6, 6.5$$

```
>>f1 = @(x) 4*x.*2.*(1-x);  % use array operators in anonymous function
>>cobweb2(f1,0.8,10)
```

For what values of λ do you think the function f is contractive near the positive fixed point?

Optional One signature of *chaos* is a sensitive dependence to initial conditions. Explore how changing the initial condition by 0.0001 changes the solutions above e.g.

```
>>f1 = @(x) 6.25*x.*^2.*(1-x);  % use array operators in anonymous function
>>cobweb2(f1,[0.7 0.7001],100)
```

Which value(s) of λ could potentially be exhibiting chaos?

The behaviour of sequences like the ones shown here is investigated more deeply in the subject 620-299 Dynamical Systems and Chaos.

2 Newtons' method

Newton's method for finding roots of Eq. 1, given by the iteration

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \tag{8}$$

can be thought of as a special kind of Picard iteration, with the function f chosen using the slope of F.

You've already solved one, because that's how Eq.7 was obtained.

2.1 Nonconvergence

Unless you're lucky, Newton's iteration is not guaranteed to be contractive, so it's possible for iterates to behave in unpredictable ways.

2.1.1 Exercise

Use the M-file `Newton295` to solve the following equations with the given initial guesses:

$$x^3 - x - 3 = 0; \quad x_0 = 0 \tag{9}$$

$$\ln(x)\exp(-x) = 0; \quad x_0 = 2 \tag{10}$$

$$1 - (1+3x)\exp(-3x) = 0; \quad x_0 = 1 \tag{11}$$

```
>>f = @(x) x.^3-x-3;
>> Newton295(f,0)
```

What happens for other initial guesses?

2.2 Fast convergence

Under favourable conditions, Newton's method converges very quickly.

2.2.1 Exercise

Compare the performance for solving

$$x^3 - 3x + 2 = 0$$

using $x_0 = 1.2$ and $x_0 = -2, 4$.

It can be proved that Newton's method converges (quickly) if you start close enough to the root. But the proof doesn't tell you how close is 'close enough'.

2.3 Heron's method for square roots

A special case is using Newton's method to find the square root of A, by solving the equation

$$x^2 - A = 0 \tag{12}$$

2.3.1 Exercise

Show that, in this case, Newton's method simplifies to

$$x_{n+1} = \frac{1}{2}(x_n + \frac{A}{x_n}) \tag{13}$$

This method was first written down by the Greek mathematician Heron of Alexandria, about 60 A.D. Try `Newton295` on Eq. 12, for $A = 2, 4$ for various (positive) initial guesses.

> **Optional**
> Explain geometrically why the sequence of iterates is monotonic. Explain geometrically why the sequence of iterates is bounded (below). This proves convergence for any positive initial guess. What properties of F in Eq. 12 guarantee that the iterates are monotonic and bounded?

Similar ideas are used to derive algorithms for *optimizing a function* i.e. finding maxima/minima. These are explored in more detail in the subject Techniques in Operations Research.