If you borrow \$500 on your credit card at 14% interest find the amounts due at the end of two years if the interest is compounded

(a) annually
(b) quarterly
(c) monthly
(d) daily
(e) hourly
(f) every second
(g) every nanosecond
(h) continuously.

(a) You owe

$500 + 500(.14) = 500(1+.14)$ after one year

$500(1+.14)(1+.14)$ after two years

(c) You owe

$500 + 500\left(\frac{.14}{12}\right) = 500\left(1+\frac{.14}{12}\right)$ after one month

$500\left(1+\frac{.14}{12}\right)\left(1+\frac{.14}{12}\right)$ after two months

$500\left(1+\frac{.14}{12}\right)^{24}$ after two years

(f) You owe

$500 + 500\,\frac{.14}{365\cdot 24\cdot 3600}$ after 1 second

and $500\left(1+\frac{.14}{365\cdot 24\cdot 3600}\right)^{2\cdot 365\cdot 24\cdot 3600}$ after 2 years.

(h) You owe $\lim_{n\to\infty}\left(1+\frac{.14}{n}\right)^{2n}\cdot 500$ after 2 years.

$$\lim_{n\to\infty} 500\left(1+\frac{.14}{n}\right)^{2n} = 500\lim_{n\to\infty} e^{\log\left(1+\frac{.14}{n}\right)^{2n}}$$

$$= 500\lim_{n\to\infty} e^{2n\log\left(1+\frac{.14}{n}\right)} = 500\lim_{n\to\infty} e^{.14\cdot 2\,\frac{\log\left(1+\frac{.14}{n}\right)}{.14/n}}$$

$$= 500\lim_{n\to\infty} e^{.28\,\frac{\log(1+.14/n)}{.14/n}}$$

Recall

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

So $\dfrac{\log(1+x)}{x} = 1 - \dfrac{x}{2} + \dfrac{x^2}{3} - \dfrac{x^3}{4} + \dfrac{x^4}{5} - \cdots$

and $\dfrac{\log\left(1+\frac{.14}{n}\right)}{\frac{.14}{n}} = 1 - \dfrac{.14}{2}\dfrac{1}{n} + \dfrac{(.14)^2}{3}\dfrac{1}{n^2} - \dfrac{(.14)^3}{3}\dfrac{1}{n^3} + \cdots$

and $\lim_{n\to\infty}\dfrac{\log\left(1+\frac{.14}{n}\right)}{\frac{.14}{n}} = 1.$

So you owe

$$500\cdot e^{.28}$$ after two years.

<u>Picard</u> <u>iteration</u> Solving $f(x)=x$.

Create a sequence (a_n):

a_1 = your choice

$a_2 = f(a_1)$,

$a_3 = f(a_2)$,

$\vdots$

Then, if $\lim_{n\to\infty} a_n = a$ (the sequence (a_n) converges)

then $f(a) = a$ (because $f(a_n) = a_{n+1}$ is very close to a_n for large enough n)

<u>Newton iteration</u> Solving $f(x)=0$.

Say the Taylor series of f at $x=a$ is

$$f(x) = f(a) + \left.\frac{df}{dx}\right|_{x=a}(x-a) + \frac{1}{2!}\left(\left.\frac{df}{dx}\right|_{x=a}\right)(x-a)^2 + \cdots$$

Then, for nice functions f; (functions that don't jump around much)

$$f(x) \approx f(a) + \left.\frac{df}{dx}\right|_{x=a}(x-a) = f(a) + f'(a)(x-a).$$

Create a sequence:

$z_0 =$ your choice

$$z_1 = \frac{-f(z_0)}{f'(z_0)} + z_0,$$

$$z_2 = \frac{-f(z_1)}{f'(z_1)} + z_1$$

$$\vdots$$

Then, if $\lim_{n\to\infty} z_n = z$ (the sequence (z_n) converges)

then

$f(z_{n+1})$ is very close to

$$f(z_n) + f'(z_n)\left(z_{n+1} - z_n\right)$$

$$= f(z_n) + f'(z_n)\left(\frac{-f(z_n)}{f'(z_n)} + z_n - z_n\right)$$

$$= 0.$$

Of course this "trick" totally fails if $f'(z_n)$ ever comes out 0 (or very very small) or if f jumps around wildly.

A sequence (a_n) is <u>contractive</u> if there exists $\alpha \in \mathbb{R}$, $\alpha \in (0,1)$, such that

$$|a_{n+1} - a_n| \leq \alpha |a_n - a_{n-1}|, \text{ for } n = 2, 3, 4, \ldots$$

If (a_n) is contractive then

$$\begin{aligned} |a_{n+1} - a_n| &\leq \alpha |a_n - a_{n-1}| \\ &\leq \alpha^2 |a_{n-1} - a_{n-2}| \\ &\leq \alpha^3 |a_{n-2} - a_{n-3}| \\ &\vdots \\ &\leq \alpha^{n-1} |a_{n-(n-2)} - a_{n-(n-1)}| \\ &= \alpha^{n-1} |a_2 - a_1| \end{aligned}$$